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BOSTON UNIVERSITY
QUESTROM SCHOOL OF BUSINESS

Dissertation

**ESSAYS IN ASSET PRICING WITH ANTICIPATIVE
INFORMATION**

by

THU TRUONG

B.Sc., McGill University, 2010

Submitted in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

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**ESSAYS IN ASSET PRICING WITH ANTICIPATIVE
INFORMATION
THU TRUONG**

Boston University, Questrom School of Business, 2015

Major Professor: Jerome Detemple, PhD ,
Professor of Finance
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ABSTRACT

This thesis focuses on private information dissemination and its impacts on financial markets. Specifically, we study issues arising when there are skilled individuals able to extract anticipative information about future prices. The first model considers a continuous time economy that is populated by informed and uninformed investors as well as active unskilled investors, and investigates the existence of noisy rational expectations equilibria and their properties. Equilibria are derived in closed form and their properties analyzed. Informed trading is found to reduce price volatility. The second model is based on the idea that besides exploiting their private information for trading purposes, informed agents might want to offer wealth management services to uninformed investors in exchange for a fee. A market for active funds emerges, and the process of anticipative information dissemination is endogenized. In this chapter, heterogeneous risk averse investors can invest in the active fund. Low risk tolerance investors are found to be strictly better off with the active fund. Fund size is not a reliable indicator of managerial skill. The market reacts to the manager's increasing risk-taking behavior by reducing the volatility and risk premium.

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List of Abbreviations

NREE Noisy Rational Expectations Equilibrium

Chapter 1

Introduction

Information is a key element in the decision making process of all the market participants, especially in an economy where some agents have private information and some do not. The ones who have private information try to take advantage of it, but at the same time want to hide this valuable information from other market participants. The uninformed agents will wish to obtain that private information. By trading in the financial market, the private information is partly disseminated to the public. Therefore, an analysis of information dissemination is very important to understand the financial markets' behaviors.

This thesis focuses on private information dissemination and its impacts on financial markets. Specifically, we study issues arising when there are skilled individuals able to extract anticipative information about future prices or dividends. There are two models in this thesis: each model studies an economy and derives the corresponding asset pricing implications. The first model considers a continuous time economy that is populated by informed and uninformed investors as well as active unskilled investors and investigates the existence of noisy rational expectations equilibria, properties of equilibrium prices and welfare. The second model is based on the idea that besides exploiting their private information for trading purposes, informed agents might want to offer wealth management services to uninformed investors in exchange for a fee. A market for active funds emerges, and the process of anticipative infor-

mation dissemination is endogenized. This second model predicts the non-monotonic relationship between fund size and managerial skill.

In the first model, the amount of private information disseminated to the market is fixed by the fraction of informed investors in the economy. Therefore, the roles of information in the market can be studied in detail. More precisely, private information circulated in the market can have two opposite effects. On the one hand, a more informative financial market seems desirable because it transmits more valuable information to uninformed investors and therefore helps to improve the allocation of resources. On the other hand, informed trading is inherently unfair because those with private information have an ultimate advantage, and thus informed trading could reduce participation, leading to a decrease in liquidity (i.e. an adverse selection effect).

This chapter examines issues surrounding the informational and allocational efficiency of the financial market, in a dynamic framework with private information about future dividends. Competitive dynamic equilibria are derived in closed form. Therefore, the behaviors of the equilibrium price and its volatility, and the risk premium properties can be studied in detail.

First, in competitive equilibria, it can be shown that weak form efficiency fails because the price is not a sufficient statistic for public information. The public information comprises both the stock price and the news on dividends.

Second, private information trading is found to stabilize the market. In fact, this private anticipative information is most valuable at the initial date. In the early stages of the economy, the price is heavily influenced by this initial information and, for this reason, does not react significantly to fundamental information. As soon as the anticipative information is disseminated in the market, all market participants learn more about the stock market evolution, so the market is more informationally

efficient. In particular, the stock price volatility is shown to decrease, relative to an equilibrium without private information.

Next, as the learning of this private anticipative information and news on dividends happens continuously, the stock market's behavior changes dynamically throughout the investment horizon. Over time, fundamental information accumulates, reducing the usefulness of the initial piece of information extracted. The impact of fundamental information on the stock price grows, thereby increasing the stock's volatility. Therefore, the volatility of the stock price increases through the dividend cycle and converges to the volatility of the fundamental as the dividend date approaches. Paradoxically, fundamental information, which accumulates through the dividend cycle and eventually announces the dividend payment, is the source of this risk increase.

Fourth, the price reactions to the underlying fundamental and to the endogenous noisy signal revealed vary through the dividend cycle. The Sharpe ratio varies stochastically through the dividend cycle. Its volatility also increases over time. Hence, the covariance between the change in the stock price and the change in the Sharpe ratio increases within a dividend cycle.

The second model extends the first model by endogenizing the dissemination process of private information. Specifically, the chapter develops a theoretical model where investors have different risk aversions to portfolio wealth fluctuations. The financial market is populated by a representative informed investor (the fund manager), uninformed investors (also referred to as public investors) and a representative noise trader. The informed investor receives a signal about the future stock price and offers investment management services to the uninformed investors. Heterogeneity in the uninformed investors' risk tolerances splits the public population into two groups, the ones with low risk tolerances optimally invest in the actively managed funds whereas the ones with high risk tolerances self-manage their portfolios (i.e. an

adverse selection effect). This segmentation in the public population endogenizes the information dissemination process and effectively affects the equilibrium asset price. Asset pricing implications are revisited.

Moreover, in a general equilibrium setting, it is found that the fund manager's skill and the fund size are not correlated. The relationship between the two is a double hump shaped curve. Indeed, an increase in managerial skill affects directly the private information precision, and indirectly the price informativeness and the economy's overall risk tolerance. An investor who knows that the manager now has higher skill has two options: he either invests in the active fund or self-manages his wealth. What happens is as follows. First, an increase in managerial skill immediately increases the private information precision and the price informativeness. However, the investors do not just evaluate the immediate benefits of the private information precision and the price informativeness but also take into account the herding behaviors of their peers. Some investors are herding; the flow is in the direction such that the marginal benefit of investing in the active fund is equal to the marginal cost of forgoing the public signal. Therefore, the equilibrium fund size is determined by the collective actions of all agents.

In fact, there are three forces simultaneously interacting in response to the manager's higher skill: private information precision, price informativeness and overall risk tolerance of the economy. The private information precision has a positive effect on the equilibrium fund size. The price informativeness, on the contrary, has a negative effect on the fund size because some of the investors are better off using the endogenous public signal instead of paying the fund manager some fee. Since the investors' risk tolerances have non-linear effects on the equilibrium risky asset prices, the risk tolerance effect (i.e. herding direction) is ambiguous. This result differentiates the actively managed fund sector from other industries whose sizes increase if

the managers' skills increase. It implies that fund size is not a reliable indicator of managerial skill.

In addition, the market reacts to the manager's increasing risk-taking behavior by reducing the market price of risk, volatility and risk premium. The market price of risk decreases because more uninformed investors switch to the active fund and only the very high risk tolerance ones self-manage their portfolios. The high average risk tolerances of the remaining public investors implies that they are able to absorb more market risk and this leads to a decrease in the market price of risk. The decrease in volatility is the consequence of an increase in price informativeness.¹ Because both market prices of risk and volatility decrease in response to an increase in the manager's risk taking level, the risk premium, which is a product of the market price of risk and the volatility, decreases accordingly.

Furthermore, the asymmetric information creates an automatic mechanism for public investors to protect themselves against the manager taking on excessive risk. In fact, the distribution of risk tolerances among public investors coupled with adverse selection set limits on the fund manager's risk-taking levels. The managers cannot implement a strategy that is excessively risky or very conservative, otherwise no one would want to invest in active funds. In particular, the distribution of risk tolerances sets investors' desired levels of the expected returns relative to the market return. On the other hand, the adverse selection induces a lower market risk premium and lower volatility if fund managers take on more risk. The combination of these elements curbs the fund manager's risk-taking incentives. If fund managers follow an extreme conservative strategy giving investors a lower expected return than they could earn from the market return by self-managing their wealth, then the active funds collapse.

¹As in Detemple, Rindisbacher & Truong (2014), where anticipative information stabilizes the stock market, an increase in price informativeness necessarily induces a decrease in equilibrium volatility.

By taking on more risk, managers can only increase fund size up to a certain extent. The reason is that when the active fund attracts enough uninformed investors, the market reacts by substantially lowering volatility and risk premium, so taking on more risk just reduces significantly the price of risk of the asset and makes the risky asset less attractive to hold, and this action is suboptimal to fund managers. Moreover, a more polarized distribution curbs the fund size. Because an increase in risk-taking level does not significantly change the average risk tolerance of the remaining public investors, the market reacts solely because of the manager's risk-taking behavior, and thus taking more risk is no longer effective.

Chapter 2

Literature Review

Classical studies pertaining to informational efficiency are based on static models. Seminal articles, identifying the determinants of efficiency in competitive markets, are those of Grossman (1976, 1978) and Grossman and Stiglitz (1980). They demonstrate the possibility, as well as the limits, of informationally efficient markets. Issues related to non-competitive behavior are examined by Hellwig (1980), Kyle (1989), Black (1992), and Leland (1992). Hellwig (1980) argues that informed investors who are aware of their price impact should not behave competitively. It resolves this apparent inconsistency, dubbed the “schizophrenia” problem, by showing that agents can no longer affect the price, in the limit competitive equilibrium, as the number of informed investors becomes large. Kyle (1989) and Black (1992) consider informed investors who explicitly account for the impact of their demands on the equilibrium price. It shows that imperfect competition resolves the schizophrenia problem. It also finds that prices are less informative with imperfect competition. Leland (1992) focuses more specifically on insider trading and on properties of equilibrium in a static model with production and monopolistic insider behavior. Among other results, it finds that private information trading increases the average stock price, decreases the stock return’s expectation and variance for the uninformed, reduces the liquidity of the market and can increase or decrease welfare.

Dynamic models with asymmetric information and competitive behavior were pioneered by Kyle (1989), Black (1992), Wang (1993, 1994). In Wang (1993, 1994),

the stock is an infinitely-lived asset that pays dividends continuously through time. Informed investors observe the state variable driving the expected future dividend. Uninformed investors do not, but they learn through dividends and prices. Noise trading injects supply uncertainty and prevents full revelation. Wang (1993) derives a competitive noisy rational expectations equilibrium (NREE). This equilibrium is stationary as the coefficients of the price process are constant. Asymmetric information is shown to increase the stock's long run risk premium. It can also increase the price volatility and enhance negative serial correlation. Asymmetric information can therefore have a destabilizing effect. Wang (1994) focuses on issues pertaining to the volume of trade in a similar, but not identical, setting. The article highlights the relation between volume and price changes. Wang (1995) study an economy with finite horizon and effects on trading volume. The effects of imperfect competition and asymmetric information on the dynamic properties of prices and liquidity are examined in Vayanos and Wang (2012). Their analysis is cast in a model with three periods. They show, in particular, that asymmetric information and imperfect competition can have opposite effects on ex-ante expected returns.¹

Albuquerque and Miao (2014) extend the competitive model of Wang (1994) by allowing for private advance information about future dividends. They also allow for a private investment opportunity. In their model, time is discrete and advance information pertains to the temporary component of the dividend paid at the next date. Agents derive utility over next period wealth. They solve for the stationary equilibrium by conjecturing a state space and a pricing rule. The stationary solution

¹ A vast microstructure literature also deals with non-competitive informed trading. Fundamental contributions are in Kyle (1985) and Glosten and Milgrom (1985). In these models, risk neutral market makers extract private information from the aggregate order flow and set the price so as to break even on average. This pricing rule does not account for the endogenous interactions between risk, price appreciation and price level. The absence of diversification benefits implies that trading is purely informational. Moreover, the price evolution is typically determined by the exogenous noise trading behavior and is locally orthogonal to fundamental risk.

is obtained up to a system of non-linear equations. The paper shows that good advanced information increases the stock price and the risk premium. It also shows that informed (resp. uninformed) investors behave as trend chasers (resp. contrarian).

The model developed in the third chapter of this thesis builds on both Wang (1993, 1994) and Albuquerque and Miao (2014). It differs in several respects. The first difference is that the analysis is not restricted to stationary equilibria. Competitive non-stationary equilibria are derived and studied. The second is that equilibria are obtained in closed form. All coefficients are explicit functions of time, reflecting the time left to the next dividend payment date. Strong timing effects are identified. The third difference is that a new solution method is introduced. The approach relies on the construction of the private information price of risk (PIPR) in the equilibrium under consideration.² The PIPR isolates the effects of private information. Its properties hint at the structure of equilibrium and can be used to formulate natural conjectures about the informational content of the stock price. The fourth difference is the nature of private information that pertains to the dividend level at the future payment date and is therefore long-lived. More precisely, information has value throughout a dividend cycle and will be used continuously for trading. The value of information, reflected in the PIPR, nevertheless changes in light of fundamental news that accumulate.

Another difference with the literature is that noise trading takes a more elaborate form in our setting. Noise traders are utility maximizing agents with bounded rationality. They hold correct beliefs conditional on the realization of the signal, but evaluate these beliefs based on unfounded rumors as opposed to factual private information. They can be viewed as unskilled active traders. Ultimately, their optimal demand behavior mimics the demand behavior of the informed, but based on

²The notion of PIPR is introduced in Detemple and Rindisbacher (2013) in the context of a portfolio selection problem with private information.

conditional beliefs evaluated at irrelevant noise. The behavioral noise trading model postulated enables us to endogenize the noise trading demand function and conduct a meaningful welfare analysis. The dynamic welfare results obtained extend the static analysis in Leland (1992).

The fourth chapter extends the third chapter by studying simultaneously two asymmetric information problems in a dynamic general equilibrium context, the first is the problem of asymmetric information about the future stock price and the second arises when the fund manager does not know the uninformed investors' characteristics (i.e. risk tolerance in this model).³ The existing literature ignores the heterogeneity in the uninformed investors' characteristics, and consequently the adverse selection in the market for actively managed funds. Hence, existing theoretical models cannot explain why, in reality, the active funds playing a substantive role in the financial markets attract a lot of investors. This model shows that the low risk tolerance investors are strictly better off with an active fund, regardless of its performance and suggests that in practice they should consider investing in the active funds.

Detemple and Rindisbacher (2013) shows that high risk tolerance investors prefer skilled funds over unskilled funds. In fact, the investors in this paper have two exclusive investment opportunities: skilled funds and unskilled funds while the investors in my thesis have two options: investing in an actively managed fund or self-managing their wealth. It would be interesting to consider a model where investors can choose between active funds, passive funds as well as their self-managed investment accounts.

Given the importance of the actively managed fund industry, many studies focus on delegated portfolio management. Existing theoretical research has focused primarily on partial equilibrium settings, particularly the agency relationship and the

³This type asymmetric information raises adverse selection problem in the active fund market. Adverse Selection is first studied by Akerlof (1970)

search for the optimal compensation contract. This literature includes Bhattacharya and Pfleiderer (1985), Starks (1987), Stoughton (1993), Ou-Yang (2003), Grinblatt and Titman (1989), Carpenter (2000), Basak, Pavlova, and Shapiro (2007). While it is true that the fund manager's investment strategy is unobservable to the investor, the principal-agent problem is not quite applicable to the delegated portfolio management context because fund managers with market timing skill are rare, while the number of uninformed public investors is enormous. Therefore, fund managers should have more bargaining power over setting a contract. Moreover, in practice, the investment management fee is set by the fund manager, not by the investors. The search for optimal contracts belongs to the normative theory rather than the positive theory of the economics of portfolio delegation. So do the asset pricing models with the agency relationship theme, Allen (2001), Brennan (1993), Gómez and Zapatero (2003), Cornell and Roll (2005), and the general equilibrium analyses of portfolio delegation in dynamic settings: Kapur and Timmermann (2005), Arora, Ju and Ou-Yang (2006), Cuoco and Kaniel (2011). For that reason, this fourth chapter focuses on the determination of the optimal fee set by the fund manager in both partial and general equilibrium settings. The introduction of proportional fee structure and adverse selection problem in the market for active funds is an attempt to contribute to the positive theory of the economics of portfolio delegation.

Another strand of related literature is the theory of the size distribution of business firms suggested by Manne (1965) and then developed by Lucas (1978) who concluded that talented managers acquire more productive factors, expanding the size of the firm. Berk and Green (2004) set up a model based on this relationship. However, the fourth chapter shows that the actively managed fund industry has a distinct feature from other industries: the size of this sector is not monotonically increasing in the fund manager skill level. An immediate implication is that the fund size might not

be a credible indicator to identify a truly skilled fund manager.

Chapter 3

Dynamic Noisy Rational Expectations Equilibria with Anticipative Information

This chapter develops a competitive equilibrium model with endogenous asymmetric information. The economy is presented in Section 3.1 and the equilibrium structure is studied in Section 3.2.

3.1 The Economy

3.1.1 Assets and Markets

There are two types of assets in the economy, a riskless asset and a risky stock. The riskless asset is a money market account paying interest at the instantaneous rate r . In the absence of intertemporal consumption, which will be assumed, the interest rate can be set at zero ($r = 0$). The risky stock pays a liquidating dividend D_T at the terminal date T . The dividend payment is the terminal value of the process,

$$dD_t = \mu^D dt + \sigma^D dW_t^D, \quad t \in [0, T]$$

where μ^D is a constant drift coefficient and σ^D is a constant and positive volatility coefficient. W^D is a Brownian motion process with filtration $\mathcal{F}_{(\cdot)}^D$, defined on a probability space $(\Omega, \mathcal{F}^D, P)$. The process D can be viewed as a fundamental factor that eventually determines the terminal dividend.

The stock trades at an endogenously determined price S . Trading takes place in

continuous time. There are no restrictions on stock holdings or borrowing.

3.1.2 Agents, Noise and Information Signal

Three groups of investors operate in the financial market, informed, uninformed and noise traders. The respective fractions of the three groups in the population are ω^i , ω^u and ω^n , with $\omega^i + \omega^u + \omega^n = 1$. Each group is treated as a homogeneous entity with a representative individual.

The (representative) informed investor is a skilled individual, able to extract information about the future stock payoff D_T . Information extraction is carried out at the initial date $t = 0$ and generates the noisy signal $G = D_T + \zeta$, where $\zeta \sim \mathcal{N}\left(0, (\sigma^\zeta)^2\right)$. Skill is measured by the precision $v_\zeta = (\sigma^\zeta)^{-2}$ of the signal. When $(\sigma^\zeta)^2$ increases, precision falls and the informational content of the signal decreases. Thus, skill decreases. In the limit, when $(\sigma^\zeta)^2 \rightarrow \infty$, the signal becomes pure noise and skill vanishes. The “informed” investor effectively becomes unskilled (uninformed).

The uninformed investor does not have extraction ability. He/she observes prices and other quantities that are in the public information set. Let $\mathcal{F}_{(\cdot)}^m$ be the public information filtration.

The noise trader is a mimicking agent. He/she tries to emulate the demand behavior of the informed agent, but on the basis of irrelevant noise as opposed to factual private information. The noise trader’s demand depends on an independent random variable ϕ . A precise description is provided below.

3.1.3 Stock Price and Information Sets

The opportunity set of investors depends on the stock price structure. In this environment, there are two sources of uncertainty, W^D associated with fundamental information and ϕ with noise trading behavior. Standard arguments can be invoked

to write any candidate price process as,

$$dS_t = \mu_t^S dt + \sigma_t^S dW_t^S, \quad S_T = D_T. \quad (3.1)$$

In this structure W^S is a Brownian motion relative to the public information filtration $\mathcal{F}_{(\cdot)}^m$. It is endogenous and, ultimately, relates to the underlying source of fundamental uncertainty W^D . The coefficients (μ^S, σ^S) of the price process are also endogenous and adapted to $\mathcal{F}_{(\cdot)}^m$. The uninformed observes the stock price, hence can retrieve the volatility coefficient from its quadratic variation. The Brownian motion $dW_t^S = (\sigma_t^S)^{-1} (dS_t - \mu_t^S dt)$ is an innovation process in their filtration. The information filtration $\mathcal{F}_{(\cdot)}^S$ generated by S is in the public information flow $\mathcal{F}_{(\cdot)}^m$. That is, $\mathcal{F}_{(\cdot)}^S \subseteq \mathcal{F}_{(\cdot)}^m$.

The information set of the informed is augmented by the private signal G . Private information is carried by the enlarged filtration $\mathcal{F}_{(\cdot)}^G \equiv \mathcal{F}_{(\cdot)}^m \vee \sigma(G)$. As private information modifies the perception of the risk-reward trade-off, the fundamental source of risk W^D is no longer Brownian motion relative to the enlarged filtration. Instead, the translated process,

$$dW_t^G = dW_t^S - \theta_t^{G|m}(G) dt \quad \text{where} \quad \theta_t^{G|m}(G) dt \equiv E[dW_t^S | \mathcal{F}_t^G]$$

becomes a Brownian motion. The translation factor $\theta_t^{G|m}(G)$ is the private information price of risk (PIPR), which is a function of the private signal G .¹ Relative to private information, the stock price evolution is $dS_t = (\mu_t^S + \sigma_t^S \theta_t^{G|m}(G)) dt + \sigma_t^S dW_t^G$. The superior information is reflected in the private information premium $\sigma_t^S \theta_t^{G|m}(G)$. Given that public information $\mathcal{F}_{(\cdot)}^m$ is endogenous, the private information premium is endogenous as well.

¹The PIPR is invariant with respect to strictly monotonic transformations of G . Indeed, the private information generated by G coincides with the information generated by $G^* = h(G)$, if $h(\cdot)$ is strictly monotone. The PIPR for G^* is $\theta_t^{G^*|m}(h^{-1}(G^*)) = \theta_t^{G|m}(G)$. As any signal with a strictly monotonic continuous distribution is a strictly monotonic transformation of a Gaussian signal, the equilibrium analysis in this paper applies for signals of this type.

3.1.4 Informed and Uninformed Preferences and Optimal Stock Demands

Throughout the chapter, superscripts i and u are used to distinguish the informed (i) from the uninformed (u) investor. Let X_t^j denote the wealth of investor j at time t , $j \in \{i, u\}$. Conditional preferences have the mean-variance structure,

$$\mathcal{U}^j(\mathcal{F}_0^j) = \begin{cases} E \left[X_T^i - \frac{1}{2\Gamma} \int_0^T d[X^i]_s \middle| \mathcal{F}_0^G \right] & \text{for } j = i \\ E \left[X_T^u - \frac{1}{2\Gamma} \int_0^T d[X^u]_s \middle| \mathcal{F}_0^m \right] & \text{for } j = u \end{cases} \quad (3.2)$$

where $[X]$ denotes the quadratic variation (realized variance) of X and Γ (resp. $1/\Gamma$) is a common absolute risk tolerance (resp. risk aversion) parameter. Preferences of the informed (resp. uninformed) are conditional on private (resp. public) information. The conditional utility functional (3.2) shows that investors care about terminal wealth X_T , but also dislike the risk $[X]_T = \int_0^T d[X]_s$, i.e., the realized variance, associated with it. The utility function depends on these two attributes.² Foundations for multiattribute preferences are in Keeney and Raiffa (1976). The ex-ante utility is $\mathcal{U}^j = E[\mathcal{U}^j(\mathcal{F}_0^j)]$ where the expectation is taken relative to the information signals in the sets $\mathcal{F}_0^j, j \in \{i, u\}$.

Investors maximize utility (3.2) subject to the dynamics of wealth,

$$dX_t^j = \begin{cases} N_t^i \left(\left(\mu_t^S + \sigma_t^S \theta_t^{G|m}(G) \right) dt + \sigma_t^S dW_t^G \right) & \text{for } j = i \\ N_t^u \left(\mu_t^S dt + \sigma_t^S dW_t^S \right) & \text{for } j = u \end{cases} \quad (3.3)$$

and the informational constraint mandating that N^j be adapted to $\mathcal{F}_{(\cdot)}^j$ for $j \in \{i, u\}$. The policy N^j represents the number of shares held. Proposition 1 describes the optimal demands.

Proposition 1 *The optimal number of shares held by the informed and uninformed*

²The preferences in (3.2) are linear in probabilities, hence time-consistent.

investors are,

$$N_t^u = \Gamma \frac{\mu_t^S}{(\sigma_t^S)^2} = \Gamma \frac{\sigma_t^S \theta_t^m}{(\sigma_t^S)^2} \quad \text{and} \quad N_t^i = \Gamma \frac{\mu_t^S + \sigma_t^S \theta_t^{G|m}(G)}{(\sigma_t^S)^2} = \Gamma \frac{\sigma_t^S (\theta_t^m + \theta_t^{G|m}(G))}{(\sigma_t^S)^2}$$

for $t \in [0, T]$, where θ^m is the price of risk for the uninformed. The informed holds more (resp. less) shares than the uninformed if and only if the private information premium $\sigma_t^S \theta_t^{G|m}(G)$ is positive (resp. negative).

Optimal stock demands have a mean-variance structure. The difference between the two investors resides in their evaluation of the expected stock return. The informed evaluates the return on the basis of private information as well as public information. The resulting expected return has two components. The first one, $\mu_t^S = \sigma_t^{S,D} \theta_t^m$, is the expected return based on public information. The second one, $\sigma_t^S \theta_t^{G|m}(G)$, is the additional premium calculated on the basis of private information. This premium is affine in the PIPR $\theta_t^{G|m}(G)$, i.e., the private information price of risk (see Detemple and Rindisbacher (2013)). The PIPR is the incremental price of risk assessed in light of information that is not revealed by public information sources. It represents the private information price of risk conditional on public information. Thus, the informed has an allocational demand, $\Gamma \theta_t^m / \sigma_t^S$, and an informational demand, $\Gamma \theta_t^{G|m}(G) / \sigma_t^S$. The uninformed has a pure allocational demand, $\Gamma \theta_t^m / \sigma_t^S$.

3.1.5 Mimicking Noise Trading and Optimal Stock Demand

The noise trader is an agent with bounded rationality, who ultimately replicates the demand of the informed, but without the benefit of observing the private signal. Instead, this investor believes in rumors, blogs and other reports that are unrelated to fundamentals underlying the stock price. Specifically, conditional beliefs are $dP^n = a_T^n(\phi) dP \equiv \exp\left(\int_0^T \theta_t^{G|m}(\phi) dW_t^S - \frac{1}{2} \int_0^T \theta_t^{G|m}(\phi)^2 dt\right) dP$ where ϕ is the realization of an independent, normally distributed random variable with mean μ^ϕ and

variance $(\sigma^\phi)^2$. The function $a_T^n(\phi)$ is a beliefs distortion capturing the departure from rationality, conditional on the realization ϕ . It corresponds to the density of the private signal, $a_T^n(\phi) = \frac{P(G \in dx | \mathcal{F}_T^m)}{P(G \in dx | \mathcal{F}_0^m)}|_{x=\phi}$, but evaluated at the noise ϕ . The informed has the same beliefs distortion, but evaluated at the private signal G . The noise trader's information is the public information filtration $\mathcal{F}_{(\cdot)}^m$.

The noise trader conditional preferences are $\mathcal{U}^n(\phi) = E^n \left[X_T^n - \frac{1}{2\Gamma} \int_0^T d[X^n]_s \middle| \mathcal{F}_0^m \right]$ where the expectation is under the beliefs P^n , Γ is an absolute risk tolerance parameter and wealth satisfies $dX_t^n = N_t^n (\mu_t^S dt + \sigma_t^S dW_t^S)$. Equivalently, conditional preferences can be written as $\mathcal{U}^n(\phi) = E \left[X_T^n - \frac{1}{2\Gamma} \int_0^T d[X^n]_s \middle| \mathcal{F}_0^G \right]_{|G=\phi}$ where the expectation is under P and information is $\mathcal{F}_0^G = \mathcal{F}_0^m \vee \sigma(G)$ evaluated at $G = \phi$. In the beliefs P^n (resp. information \mathcal{F}_0^G evaluated at ϕ) the stock price evolves according to $dS_t = \left(\mu_t^S + \sigma_t^S \theta_t^{G|m}(\phi) \right) dt + \sigma_t^S dW_t^\phi$ where W^ϕ is a P^n -Brownian motion (resp. $\mathcal{F}_0^{G|\phi}$ -Brownian motion). The stock price of risk is believed to be $\theta_t^\phi \equiv \theta_t^m + \theta_t^{G|m}(\phi)$. Ex-ante utility is $\mathcal{U}^n = E[\mathcal{U}^n(\phi)]$ where the expectation is over the random variable ϕ .

Proposition 2 *The optimal number of shares held by the noise trader is,*

$$N_t^n = \Gamma \frac{\sigma_t^S \left(\theta_t^m + \theta_t^{G|m}(\phi) \right)}{(\sigma_t^S)^2} \quad (3.4)$$

for $t \in [0, T]$, where θ_t^m is the uninformed price of risk and $\theta_t^{G|m}(\phi)$ is a speculative premium/discount reflecting the departure from rationality. The noise trader holds more (resp. less) shares than the uninformed if and only if the speculative premium $\sigma_t^S \theta_t^{G|m}(\phi)$ is positive (resp. negative).

The optimal noise trading demand has two parts. The first part, $\Gamma \theta_t^m / \sigma_t^S$, is the usual mean-variance demand of an uninformed rational agent. This part reflects an allocational trading motive. The second part, $\Gamma \theta_t^{G|m}(\phi) / \sigma_t^S$, is a speculative demand associated with an informational signal consisting of pure noise. In the end, the noise

trader demand mimics the demand of the informed. It effectively corresponds to the demand of an investor with randomized beliefs, i.e., an unskilled active investor.

Remark 3 *The combined demand of the informed and the noise trader, called the complementary demand, is,*

$$N_t \equiv \omega^i N_t^i + \omega^n N_t^n = \Gamma \frac{\omega \mu_t^S + \sigma_t^S \left(\omega^i \theta_t^{G|m}(G) + \omega^n \theta_t^{G|m}(\phi) \right)}{(\sigma_t^S)^2}$$

where $\omega = \omega^i + \omega^n$. The complementary demand is an affine function of the weighted average price of risk (WAPR) $\Theta_t(G, \phi; \omega^i, \omega^n) \equiv \omega^i \theta_t^{G|m}(G) + \omega^n \theta_t^{G|m}(\phi)$. If the PIPR is also an affine function, the complementary demand depends on $\Theta_t(G, \phi; \omega^i, \omega^n) = \Theta_t(Z; \omega)$, which is a function of the signal $Z \equiv \omega^i G + \omega^n \phi$ and is parametrized by the combined population weight $\omega = \omega^i + \omega^n$.

3.1.6 Equilibrium

A rational expectations equilibrium (REE) for the economy under consideration is a triplet of demands (N^u, N^i, N^n) and a price process $dS_t = \mu_t^S dt + \sigma_t^S dW_t^S$, $S_T = D_T$, such that (i) Individual rationality: N^j is optimal for agent $j \in \{u, i, n\}$, and (ii) Market clearing: $\omega^u N^u + \omega^i N^i + \omega^n N^n = 1$.

The REE is noisy (NREE) if the informed and uninformed filtrations differ, $\mathcal{F}_{(\cdot)}^u \subset \mathcal{F}_{(\cdot)}^i$. The equilibrium is a competitive NREE if all agents take the price process as given when expressing their optimal demands. It is a monopolistic NREE if the informed agent takes the price impact of his/her trades into account when calculating the optimal demand function.

3.2 Competitive Noisy Rational Expectations Equilibrium

The competitive NREE is described in Section 3.2.1. Properties of the PIPR and the WAPR are examined in Section 3.2.2. Price and return properties are discussed in Section 3.2.3. Properties of the market depth measure and stock holdings are outlined

in Section 3.2.4.

3.2.1 Competitive Equilibrium Structure

In order to present the main result, define the combined share of the informed and the noise trader $\omega = \omega^i + \omega^n$ and the functions of time,

$$\alpha(t) = \frac{1 - \kappa_t \omega}{H(t)} \sigma^D, \quad \beta(t) = -\omega \frac{1 - \kappa_t \omega^i}{H(t)} \sigma^D, \quad \kappa_t = \frac{\omega^i H(t)}{M(t)} \quad (3.5)$$

$$\gamma(t) = -\omega \frac{(1 - \kappa_t \omega^i) \mu^D (T - t) - \omega^n \kappa_t \mu^\phi}{H(t)} \sigma^D, \quad \lambda(t, s) = \frac{\omega^i (\sigma^D)^2 (s - t)}{M(t)}, \quad s \in [t, T] \quad (3.6)$$

$$H(t) = (\sigma^D)^2 (T - t) + (\sigma^\zeta)^2, \quad M(t) = (\omega^i)^2 H(t) + (\omega^n)^2 (\sigma^\phi)^2. \quad (3.7)$$

The function $H(t) = \text{Var}(G|\mathcal{F}_t^D)$ is the conditional variance of the private signal G given fundamental information at time t . The function $M(t) = \text{Var}(Z|\mathcal{F}_t^D)$ is the conditional variance of an endogenous signal $Z \equiv \omega^i G + \omega^n \phi$ given fundamental information at time t . The coefficients $\kappa_t = \frac{\text{COV}(G, Z|\mathcal{F}_t^D)}{\text{VAR}(Z|\mathcal{F}_t^D)}$ and $\lambda(t, s) = \frac{\text{COV}(D_s, Z|\mathcal{F}_t^D)}{\text{VAR}(Z|\mathcal{F}_t^D)}$ are regression coefficients. The next proposition presents the NREE.

Proposition 4 *A competitive NREE exists. The equilibrium stock price is,*

$$S_t = A(t)D_t + B(t)Z + F(t) \quad \text{where} \quad Z = \omega^i G + \omega^n \phi \quad (3.8)$$

and,

$$A(t) = \left(\frac{H(T)}{H(t)} \right)^\omega \left(\frac{M(T)}{M(t)} \right)^{1-\omega} \quad (3.9)$$

$$B(t) = \lambda(t, T) + \sigma^D \left(\int_t^T A(s) (\alpha(s) + \beta(s)\lambda(t, s)) ds \right) \quad (3.10)$$

$$F(t) = A(t) \mu^D (T - t) - \frac{(\sigma^D)^2}{\Gamma} \int_t^T A(s)^2 ds + \sigma^D \int_t^T A(s) \gamma(s) ds - \omega^n I(t) \mu^\phi \quad (3.11)$$

$$I(t) = \lambda(t, T) + \sigma^D \int_t^T A(s) \beta(s) \lambda(t, s) ds \quad (3.12)$$

with $(\alpha, \beta, \gamma, \lambda)$ as defined in (3.5)-(3.7). The coefficients of the equilibrium stock price process (3.1) are,

$$\mu_t^S = \frac{(\sigma_t^S)^2}{\Gamma} - \sigma_t^S \Theta_t(Z; \omega), \quad \sigma_t^S = A(t) \sigma^D \quad (3.13)$$

$$\Theta_t(Z; \omega) = \alpha(t) Z + \beta(t) D_t + \gamma(t) \quad (3.14)$$

where $\Theta_t(Z; \omega) \equiv \omega^i \theta_t^{G|m}(G) + \omega^n \theta_t^{G|m}(\phi)$ is the endogenous WAPR. Innovations in the uninformed filtration are $dW_t^S = dW_t^D - \theta_t^{D|m} dt$ with,

$$\theta_t^{D|m} = \frac{E[dW_t^D | \mathcal{F}_t^m]}{dt} = \frac{\omega^i \sigma^D}{M(t)} (Z - \omega^i (D_t + \mu^D (T - t)) - \omega^n \mu^\phi). \quad (3.15)$$

The evolution of the stock price in the public information is given by (3.1) where W^S is an $\mathcal{F}_{(\cdot)}^m = \mathcal{F}_{(\cdot)}^{D,Z}$ -Brownian motion.

The competitive equilibrium price in (3.8) is an affine function of the fundamental D and of the random variable Z . This random variable is a noisy translation of the private information signal G . It provides anticipative information about the terminal dividend, but is less informative than the private signal. Both the price S and the fundamental D are in the public information set $\mathcal{F}_{(\cdot)}^m$. It follows that Z is publicly observed as well. Thus, $Z \in \mathcal{F}_{(\cdot)}^m$ and $\mathcal{F}_{(\cdot)}^{D,Z} \subseteq \mathcal{F}_{(\cdot)}^{D,S} \subseteq \mathcal{F}_{(\cdot)}^m$. Conversely, the pair (D, Z) reveals the price S , i.e., $\mathcal{F}_{(\cdot)}^S \subseteq \mathcal{F}_{(\cdot)}^{D,Z}$. Thus, $\mathcal{F}_{(\cdot)}^{D,S} = \mathcal{F}_{(\cdot)}^{D,Z} \subseteq \mathcal{F}_{(\cdot)}^m$.

In equilibrium, the uninformed extracts the noisy signal Z from the pair (D, S) . The uninformed also observes the complementary aggregate demand function $\omega^i N_t^i + \omega^n N_t^n$, described in Remark 3. At equilibrium, the complementary demand is also affine in D and Z . If therefore fails to reveal any information beyond what is already contained in (D, S) . In the end, the equilibrium public information set consists of the pair (D, Z) . That is, $\mathcal{F}_{(\cdot)}^{D,S} = \mathcal{F}_{(\cdot)}^{D,Z} = \mathcal{F}_{(\cdot)}^m$. The equilibrium uninformed filtration

is $\mathcal{F}_{(\cdot)}^u = \mathcal{F}_{(\cdot)}^m = \mathcal{F}_{(\cdot)}^{D,S} = \mathcal{F}_{(\cdot)}^{D,Z}$. The equilibrium informed filtration is strictly more informative, $\mathcal{F}_{(\cdot)}^i = \mathcal{F}_{(\cdot)}^G = \mathcal{F}_{(\cdot)}^m \vee \sigma(G) \supset \mathcal{F}_{(\cdot)}^m = \mathcal{F}_{(\cdot)}^u$. The equilibrium is a noisy rational expectations equilibrium.

In this competitive NREE, the stock price S_t is not a sufficient statistic for public information. In fact, $\sigma(S_t) \subset \mathcal{F}_t^m = \mathcal{F}_t^{D,Z}$, where the inclusion is strict. Weak form efficiency therefore fails. The pairs (D, Z) or (D, S) are needed to summarize the public information set. Fundamental information plays a crucial role for the evaluation of future opportunities and the determination of optimal demands.

Remark 5 (*Limit economy with small informed*) Consider the limit economy with an infinitesimal informed population ($\omega^i \rightarrow 0$ and $\omega^u \rightarrow 1 - \omega^n$). The limit equilibrium is,

$$\begin{aligned} S_t^{si} &= A^{si}(t)D_t + B^{si}(t)Z^{si} + F^{si}(t), & Z^{si} &= \omega^n \phi \\ \mu_t^{S,si} &= \frac{(\sigma_t^{S,si})^2}{\Gamma} - \sigma_t^{S,si} \Theta_t^{si}(Z^{si}; \omega^n), & \sigma_t^{S,si} &= A^{si}(t) \sigma^D \\ \Theta_t^{si}(Z^{si}; \omega^n) &= \alpha^{si}(t) Z^{si} + \beta^{si}(t) D_t + \gamma^{si}(t) \end{aligned}$$

where $(A^{si}, B^{si}, F^{si}, \alpha^{si}, \beta^{si}, \gamma^{si})$ are defined in (2)-(4). The limit WAPR is $\Theta_t^{si}(Z^{si}; \omega^n) = \omega^n \theta_t^{G|m,si}(\phi)$. Innovations in the uninformed filtration vanish $dW_t^S = dW_t^D$ because $\theta_t^{D|m} \rightarrow 0$ when $\omega^i \rightarrow 0$. The limit equilibrium fails to reveal any private information. If, in addition, there is no mimicking investor ($\omega^i, \omega^n \rightarrow 0$), the equilibrium collapses to a no-trade equilibrium where,

$$S_t^{si,0} = D_t + \mu^D (T - t) - \frac{(\sigma^D)^2}{\Gamma} (T - t), \quad \sigma_t^{S,si,0} = \sigma^D, \quad \mu_t^{S,si,0} = \frac{(\sigma^D)^2}{\Gamma}.$$

Stock price volatilities in the economy of Proposition 4 and the two limit economies rank as $\sigma_t^S < \sigma_t^{S,si} < \sigma_t^{S,si,0} = \sigma^D$ for $t < T$. As the payment date approaches the volatilities converge, $\lim_{t \rightarrow T} \sigma_t^S = \lim_{t \rightarrow T} \sigma_t^{S,si} = \lim_{t \rightarrow T} \sigma_t^{S,si,0} = \sigma^D$. Informed trading increases the informational efficiency of the market. It also stabilizes the market by reducing the stock's exposure to fundamental shocks and the associated price volatility.

Remark 6 (*Limit economy with small uninformed*) Consider the limit economy with an infinitesimal uninformed population ($\omega^i \rightarrow 1 - \omega^n$ and $\omega^u \rightarrow 0$). The limit equi-

librium is,

$$\begin{aligned}
S_t^{su} &= A^{su}(t) D_t + B^{su}(t) Z^{su} + F^{su}(t), & Z^{su} &= (1 - \omega^n) G + \omega^n \phi \\
\mu_t^{S,su} &= \frac{(\sigma_t^{S,su})^2}{\Gamma} - \sigma_t^{S,su} \Theta_t^{su}(Z^{su}; 1), & \sigma_t^{S,su} &= A^{su}(t) \sigma^D \\
\Theta_t^{su}(Z^{su}; 1) &= \alpha^{su}(t) Z^{su} + \beta^{su}(t) D_t + \gamma^{su}(t)
\end{aligned}$$

where the functions $(A^{su}, B^{su}, F^{su}, \alpha^{su}, \beta^{su}, \gamma^{su})$ are defined in (5)-(10). If, in addition, there is no mimicking investor ($\omega^i \rightarrow 1, (\omega^u, \omega^n) \rightarrow 0$), the equilibrium collapses to a no-trade equilibrium where

$$\begin{aligned}
S_t^{su,0} &= A^{su}(t) D_t + B^{su,0}(t) G + F^{su,0}(t), & Z^{su} &= G \\
\mu_t^{S,su,0} &= \frac{(\sigma_t^{S,su,0})^2}{\Gamma}, & \sigma_t^{S,su,0} &= \sigma_t^{S,su} = A^{su}(t) \sigma^D, & \Theta_t^{su}(Z^{su}; 1) &= 0
\end{aligned}$$

with $(A^{su}, B^{su,0}, F^{su,0})$ as defined in (11)-(12). The pair $(D, S^{su,0})$, in the limit economy, is fully revealing. Stock price volatilities in the three equilibria rank as $\sigma_t^{S,su} = \sigma_t^{S,su,0} < \sigma_t^S < \sigma^D$ for $t < T$. As the payment date approaches, $\lim_{t \rightarrow T} \sigma_t^{S,su} = \lim_{t \rightarrow T} \sigma_t^{S,su,0} = \lim_{t \rightarrow T} \sigma_t^S = \sigma^D$. Equilibrium prices in economies with small uninformed (large informed) populations are less sensitive to fundamental shocks and have lower volatility.

3.2.2 PIPR and WAPR Properties

To provide further insights about the structure of equilibrium, it is instructive to start with the PIPR. The PIPR is the (negative of the) instantaneous volatility of the growth rate of the conditional density of the private information signal given public information. In equilibrium, with $\mathcal{F}_t^m = \mathcal{F}_t^{D,Z}$,

$$\theta_t^{G|m}(G) = \text{vol} \left(\frac{dp_t^G(G)}{p_t^G(x)} \right) = \frac{G - \mu_t^{G|D,Z}}{(\sigma_t^{G|D,Z})^2} \text{vol} \left(\mu_t^{G|D,Z} \right) = \frac{G - \mu_t^{G|D,Z}}{(\sigma_t^{G|D,Z})^2} (1 - \kappa_t \omega^i) \sigma^D.$$

In the model under consideration, given the linearity of the endogenous signal Z revealed, the conditional density is normal. The conditional mean alone depends on

the dividend. The conditional variance is a function of time. The PIPR therefore reduces to the volatility of the conditional mean suitably normalized. It is affine in the private signal. As noted in Remark 3, it follows that the WAPR becomes $\Theta_t(G, \phi; \omega) \equiv \Theta_t(Z; \omega)$ and that the complementary demand is an affine function of $\Theta_t(Z; \omega)$. The equilibrium risk premium inherits this affine structure. Moreover, the equilibrium complementary demand, being affine in $\Theta_t(Z; \omega)$, also reveals the signal $Z = \omega^i G + \omega^n \phi$.

The next corollary describes the behavior of the endogenous PIPR.

Corollary 7 *The equilibrium PIPR is,*

$$\theta_t^{G|m}(G) = \frac{G - \mu_t^{G|D,Z}}{\left(\sigma_t^{G|D,Z}\right)^2} (1 - \kappa_t \omega^i) \sigma^D = \alpha_1(t) G + \alpha_2(t) Z + \beta_0(t) D_t + \gamma_0(t)$$

$$\alpha_1(t) \equiv \frac{\sigma^D}{H(t)}, \quad \alpha_2(t) \equiv -\frac{\kappa_t \sigma^D}{H(t)} = -\frac{\omega^i \sigma^D}{M(t)}, \quad \beta_0(t) = \frac{\beta(t)}{\omega}, \quad \gamma_0(t) = \frac{\gamma(t)}{\omega}$$

where $\omega = \omega^i + \omega^n$ and $\beta(t), \gamma(t)$ are defined in (3.5)-(3.7). The coefficients $\alpha_1(t)$, $\alpha_2(t)$ and $\beta(t)$ are the sensitivities with respect to the private signal G , the endogenous public signal Z and the fundamental D_t . The coefficient $\gamma(t)$ is a translation factor. The following properties hold,

(i) *Sensitivity to information:* $\alpha_1(t) > 0, \alpha_2(t) < 0, \beta(t) < 0$.

(ii) *Dynamic behavior:*

$$(ii-1) \quad \frac{\partial \alpha_1(t)}{\partial t} > 0, \quad \frac{\partial \alpha_2(t)}{\partial t} < 0, \quad \frac{\partial \beta(t)}{\partial t} < 0$$

$$(ii-2) \quad \frac{\partial \gamma(t)}{\partial t} > 0 \text{ if and only if } H(t) < H^+ \text{ with } H^+ \text{ as defined in (13)-(15).}$$

(iii) *Population effects (informed to noise trader ratio):* Fix ω and let $s = \omega^i / \omega^n$ vary. Then,

$$(iii-1) \quad \frac{\partial \alpha_1(t)}{\partial s} = 0, \quad \frac{\partial \beta(t)}{\partial s} > 0$$

$$(iii-2) \quad \frac{\partial \alpha_2(t)}{\partial s} > 0 \text{ if and only if } H(t) > \frac{2s+1}{s^2} (\sigma^\phi)^2$$

$$(iii-3) \quad \frac{\partial \gamma(t)}{\partial s} > 0 \text{ if and only if } 2s (\sigma^\phi)^2 \mu^D (T-t) > \left(-s^2 H(t) + (\sigma^\phi)^2\right) \mu^\phi.$$

(iv) *Bias effects:* $\frac{\partial \alpha_1(t)}{\partial \mu^\phi} = \frac{\partial \alpha_2(t)}{\partial \mu^\phi} = \frac{\partial \beta(t)}{\partial \mu^\phi} = 0, \quad \frac{\partial \gamma(t)}{\partial \mu^\phi} < 0$.

The reaction of the equilibrium PIPR to news is intuitive. Indeed, a larger private signal indicates a greater terminal dividend, thus provides more valuable information.

In contrast public information, be it endogenous or exogenous, reduces the local value of private information.

The evolution of these sensitivities over time is also intuitive. The reaction to private information $\alpha_1(t)$ is tamed by the unconditional variance of the signal $H(t)$ in the denominator. Over time, the informed observes the fundamental and updates the content of the private signal. Effectively, the residual private information is $G - D_t$. This residual signal becomes more informative over time, as uncertainty resolves, thereby enhancing the value of information. For the same reason, the precision of the endogenous public signal increases. This reduces the (negative) sensitivity of the PIPR to the endogenous signal, which decreases the value of private information. The reaction to fundamental information reflects the same effect. Its decrease contributes to a further reduction in the value of information.

Population effects can be traced to the informational content of the endogenous public signal which depends on the relative fraction s of informed to noise trader. When s increases, endogenous information becomes more precise. This decreases both sensitivities,

$$\alpha_2(t) \equiv -\frac{\kappa_t}{H(t)}\sigma^D = -\frac{\omega^i}{M(t)}\sigma^D = -\frac{1}{\omega} \frac{1 + \frac{1}{s}}{H(t) + \frac{1}{s^2}(\sigma^\phi)^2}\sigma^D$$

$$\beta_0(t) \equiv -\frac{1 - \kappa_t\omega^i}{H(t)}\sigma^D = -\frac{\frac{1}{s^2}(\sigma^\phi)^2}{H(t) \left(H(t) + \frac{1}{s^2}(\sigma^\phi)^2 \right)}\sigma^D$$

(denominator effects), which become more negative. At the same time, the covariance between the endogenous signal and private information decreases (numerator effect), which increases the sensitivities. In the case of $\alpha_2(t)$, the second effect dominates under the condition stated. For $\beta_0(t)$, it always dominates.

The impact of the bias is through the conditional mean of the private signal. A higher bias increases the conditional mean, leading to a reduction in the PIPR.

The WAPR is closely related to the PIPR and inherits most of its properties.

Corollary 8 *The equilibrium WAPR is given by (3.14). The coefficients $\alpha(t)$ and $\beta(t)$ are the sensitivities with respect to the endogenous public signal and the fundamental information. The coefficient $\gamma(t)$ is a translation factor. The properties of $(\beta(t), \gamma(t))$ are the same as those of $(\beta_0(t), \gamma_0(t))$ in Corollary 7. The behavior of $\alpha(t)$ differs in the following respects,*

- (i) *Sensitivity to information: $\alpha(t) > 0$ if and only if $(\sigma^\phi)^2 > sH(t)$.*
- (ii) *Dynamic behavior: $\alpha(t)$ increases with time if and only if $\kappa_t^2 < 1/\omega^i\omega$.*
- (iii) *Population effects: $\alpha(t)$ increases with s if and only if $H(t) > \frac{2s+1}{s^2}(\sigma^\phi)^2$.*

The behavior of $\alpha(t) = \alpha_1(t) + \alpha_2(t)\omega$ is more intricate because $\alpha_1(t), \alpha_2(t)$ have different, sometimes opposite properties. The evolution of $\alpha(t)$ over time is especially noteworthy. If $\omega^i\omega\kappa_0^2 < 1$, the coefficient increases over time. If $\omega^i\omega\kappa_0^2 > 1$ and $\omega^i\omega\kappa_T^2 < 1$, it initially decreases, then increases. If $\omega^i\omega\kappa_0^2 > 1$ and $\omega^i\omega\kappa_T^2 > 1$, it decreases throughout. The possibility of a U -shaped pattern reflects conflicting effects on $\alpha_1(t)$ and $\alpha_2(t)$. Under the conditions stated, the decrease in $\alpha_2(t)$ dominates early on, then is overtaken by the increase in $\alpha_1(t)$. An illustration is in Figure 3.1. The figure presents the dynamic behavior of $\alpha(t)$ for $t \in [0, T]$. Parameter values are $T = 1$, $\sigma^D = 0.1$, $\sigma^\phi = STD[G]$, $\sigma^\zeta = 0.32$. The weight of informed is $\omega^i = \frac{1}{2}$. The weight of mimicking noise traders varies between $\omega^n = 0.05$ (left panel), $\omega^n = 0.22$ (middle panel) and $\omega^n = 0.25$ right panel.

3.2.3 Price and Return Properties

Fundamental information accumulates with the passage of time, providing more precise estimates of the next dividend payment. Information accumulation affects the properties of equilibrium. The next corollary describes the dynamic behavior of the price and the return components.

Corollary 9 *The stock price sensitivity to the fundamental (resp. the endogenous public signal) increases (resp. decreases) over time. The volatility of the stock price,*

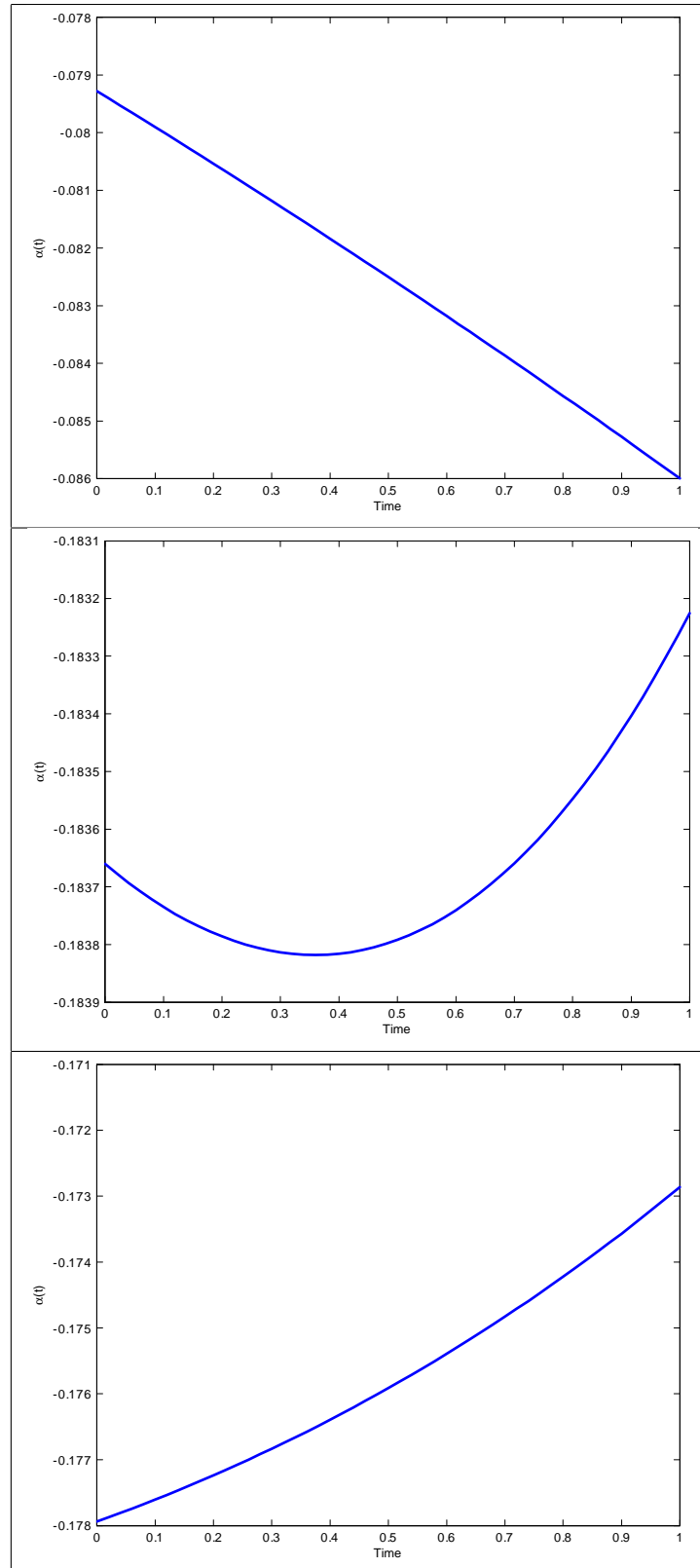


Figure 3.1: Sensitivity of WAPR to private information

$\sigma_t^S = A(t) \sigma^D$, increases over time. The minimal and maximal volatility values are obtained at the initial and terminal dates,

$$\sigma_0^S = A(0) \sigma^D = \left(\frac{H(T)}{H(0)} \right)^\omega \left(\frac{M(T)}{M(0)} \right)^{1-\omega} \sigma^D, \quad \lim_{t \rightarrow T} \sigma_t^S = A(T) \sigma^D = \sigma^D.$$

The stock's price of risk $\mu_t^S / \sigma_t^S = A(t) \sigma^D / \Gamma - (\alpha(t) Z + \beta(t) D_t + \gamma(t))$ becomes more sensitive to the fundamental over time (i.e., $-\beta(t) > 0$ increases for all $t \in [0, T]$). Its sensitivity with respect to the endogenous public signal increases at date t if and only if $\omega^i \omega \kappa_t^2 < 1$ (i.e., $-\alpha(t)$ increases if $\omega^i \omega \kappa_t^2 < 1$).

At the initial date, the uninformed extracts the noisy signal Z from the price. This information is most valuable when there is no other source of information, i.e., at the initial date. In the early stages of the economy, the price is heavily influenced by this initial information and, for this reason, does not react significantly to fundamental information. Over time, fundamental information accumulates, reducing the usefulness of the initial piece of information extracted. The impact of fundamental information (resp. the endogenous noisy signal) on the stock price grows (resp. decreases), thereby increasing the stock's volatility.

The behavior of the price of risk is more intricate. As for the stock price, the sensitivity to fundamental information increases. The volatility of the price of risk therefore increases over time. The sensitivity with respect to the endogenous public signal can exhibit three types of patterns. If $\omega^i \omega \kappa_0^2 < 1$, it decreases over time. If $\omega^i \omega \kappa_0^2 > 1$ and $\omega^i \omega \kappa_T^2 < 1$, it initially increases, then decreases. If $\omega^i \omega \kappa_0^2 > 1$ and $\omega^i \omega \kappa_T^2 > 1$, it increases throughout. The possibility of an \cap -shaped pattern reflects the U -shaped behavior of the WAPR. Figure 3-2 illustrates the price of risk and volatility behaviors. The figure presents the dynamic behavior of σ_t^S and $\beta(t)$ for $t \in [0, T]$. Parameter values are $T = 1$, $\sigma^D = 0.1$, $\sigma^\phi = STD[G]$, $\sigma^\zeta = 0.5$. The weight of informed and mimicking noise traders are $\omega^i = \omega^n = \frac{1}{3}$.

The impact of risk attitudes is outlined next.

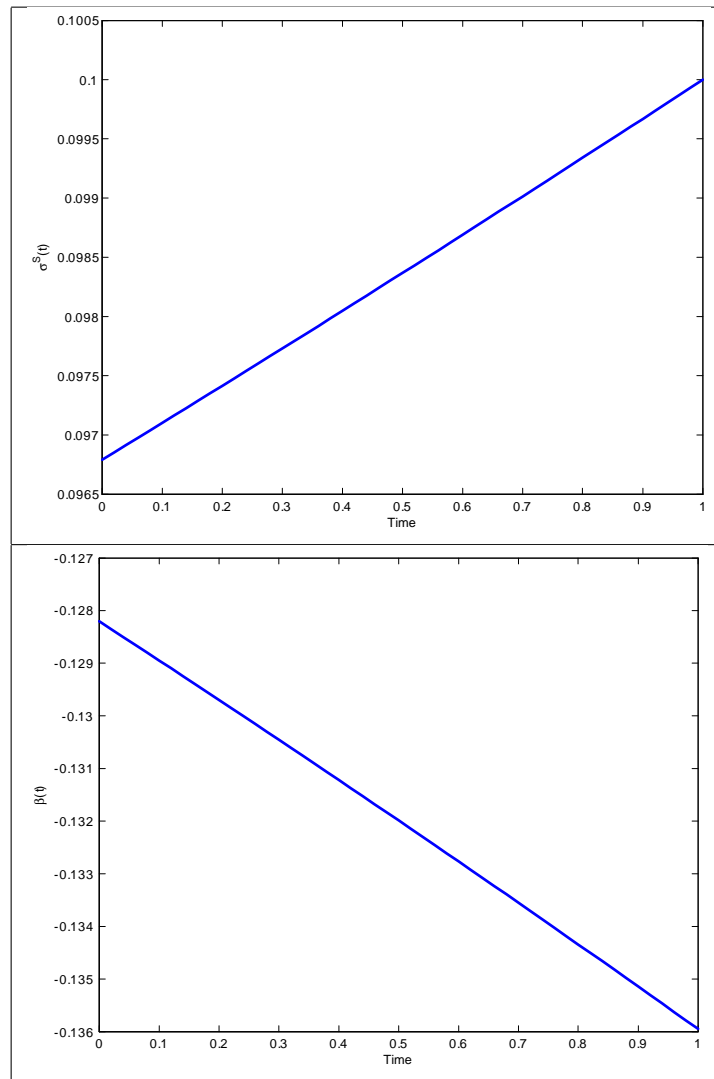


Figure 3.2: Stock volatility and sensitivity of WAPR to fundamental information

Corollary 10 *The stock price is an increasing function of risk tolerance, but its sensitivity coefficients with respect to fundamental information and to the noisy signal do not depend on it. Likewise, the volatility of the stock price is not affected by risk tolerance. The stock's risk premium is a decreasing function of risk tolerance.*

An increase in risk tolerance promotes an increase in the demand for the stock, which increases value. As shown by expression (3.11) for $F(t)$, risk tolerance effectively acts on the risk discount embedded in the stock price. When risk tolerance increases, the willingness to bear risk increases, reducing the price discount required to hold the asset. The absence of an impact on the coefficients $(A(t), B(t))$ capturing the price sensitivity to the information sources (D_t, Z) , follows from the mean-variance structure of the demand functions and the assumption of common risk attitudes across investors. Under these circumstances, the aggregate demand function is an affine function of the WAPR, that carries information and is unrelated to risk attitudes. The stock price inherits this behavior. It depends on information through the WAPR, unaffected by risk attitudes. Moreover, the absence of an impact on the sensitivity $A(t)$ with respect to the fundamental implies that the volatility of the stock price is not affected by risk attitudes either.

Because aggregate demand has a mean-variance form, the risk premium is also linear in the PIPR. The risk premium is determined by the return variance per unit risk tolerance adjusted by a discount related to the WAPR. Given that the variance of the stock price and the WAPR do not depend on risk attitudes the result stated follows.

The last corollary in this section reports the effects of variations in the population of investors and in the noise trading bias.

Corollary 11 *Suppose that the ratio of informed to noise trader, $s \equiv \omega^i/\omega^n$, increases, but that their combined fraction in the population, $\omega = \omega^i + \omega^n$, stays the same. Under this scenario, the sensitivity of the stock price with respect to fundamental information and its volatility both decrease. The stock's risk premium can increase*

or decrease. If the noise trading bias $\mu^\phi - E[G]$ (i.e., μ^ϕ) increases, the stock price decreases. The stock volatility is not affected. The stock's risk premium increases.

When the fraction of informed to noise trader s increases, the information extracted from the price becomes more precise. This tames the response to other sources of information such as the fundamental. The volatility of the stock price, which is entirely driven by the volatility of the fundamental, inherits this behavior. In contrast, the stock risk premium can increase or decrease because of the conflicting effects on the coefficients of the WARP. The effects of noise trading bias are straightforward and follow from the behavior of the non-stochastic component of the WARP.

3.2.4 Market Depth and Investor Strategies

Market Depth Properties

Market depth seeks to capture the impact of trading on the price. It is typically measured by the inverse of the coefficient of the regression of the stock price on the complementary demand function (Kyle (1985)). Properties of market depth are described next,

Corollary 12 *Market depth m is given by,*

$$m(t) \equiv \left(\frac{d[S_t, N_t]}{d[N_t, N_t]} \right)^{-1} = \frac{\frac{\omega^u \Gamma}{\sigma^D} \frac{\beta(t)}{A(t)} \sigma_D \times \frac{\omega^u \Gamma}{\sigma^D} \frac{\beta(t)}{A(t)} \sigma_D}{\frac{\omega^u \Gamma}{\sigma^D} \frac{\beta(t)}{A(t)} \sigma_D \times A(t) \sigma_D} = \frac{\omega^u \Gamma \beta(t)}{A^2(t) \sigma_D}. \quad (3.16)$$

Market depth is negative, and increases over time if and only if $\omega > 1/2$. Under this condition, its minimal and maximal values are reached at the initial and terminal dates,

$$\begin{aligned} m(0) &= -\frac{\omega^u \Gamma \omega (\omega^n)^2 (\sigma^\phi)^2}{H(0) M(0)} \left(\frac{H(0)}{H(T)} \right)^{2\omega} \left(\frac{M(0)}{M(T)} \right)^{2(1-\omega)} \\ m(T) &= -\frac{\omega^u \Gamma \omega (\omega^n)^2 (\sigma^\phi)^2}{H(T) M(T)}. \end{aligned}$$

It also decreases with risk tolerance Γ and increases with the fraction s of informed. Market depth is not related to the bias component μ^ϕ .

Market depth is negative because the covariance between the price change and the change in the combined demand of the informed and the noise trader is negative. The passage of time has two effects on depth. On the one hand, it increases the volatility of the stock price, which increases the covariance between the stock price change and the demand change. On the other hand, it has a negative effect on the volatility of the complementary demand through the coefficient $\beta(t)$, which becomes more negative. The trade-off between these two opposite effects is determined by the fraction of informed and noise trader in the total population. When this fraction is greater than half, the first effect dominates, leading to an increasing market depth over time, i.e., a market depth that becomes less negative. When the informed and noise trader are a majority, the price effect is dominated by the demand effect. When the informed and the noise trader form a minority, the price impact is sufficiently important to offset the demand effect.

The behavior with respect to the other quantities such as risk tolerance and the informed-to-noise trader ratio is monotone. The latter increases, because the volatility of the stock price decreases while the volatility of the complementary demand increases. Both effects contribute to an increase in market depth.

Remark 13 *Collin-Dufresne and Fos (2013) generate time-varying market depth by extending Kyle (1985) to more general processes for exogenous noise trading. Their time-varying measures of liquidity are supported by their empirical findings (Collin-Dufresne and Fos (2014)) that, in contrast to the predictions of standard microstructure models, market depth can increase with more informed trading. Market depth in the present model is tied to the underlying fundamental. It is time-varying and can also increase with informed trading.*

Momentum and Reversal Strategies

The next corollary describes the investment strategies of the three groups of agents.

Corollary 14 *Let $N_t^{i,G} = \Gamma \theta_t^{G|m}(G) / \sigma_t^S$ be the private information component of the informed demand. The optimal portfolio policy of the informed (resp. uninformed) is a contrarian (resp. momentum) strategy,*

$$\begin{aligned} \frac{d[N^u, S]_t}{dt} &= \frac{\Gamma}{\sigma^D} \frac{d[\theta^m, D]_t}{dt} = -\frac{\Gamma}{\sigma^D} \frac{d[\Theta_t(Z; \omega), D]_t}{dt} = -\Gamma \beta(t) > 0 \\ \frac{d[N^{i,G}, S]_t}{dt} &= \frac{\Gamma}{\sigma^D} \frac{d[\theta^{G|m}(G), D]_t}{dt} = -\frac{\Gamma \sigma^D}{H(t)} \frac{d[\mu^{G|D,Z}, W^D]_t}{dt} = \frac{\Gamma}{\omega} \beta(t) < 0 \\ \frac{d[N^i, S]_t}{dt} &= \frac{d[N^u + N^{i,G}, S]_t}{dt} = \Gamma \left(-1 + \frac{1}{\omega} \right) \beta(t) < 0. \end{aligned}$$

The mimicking noise trader pursues a contrarian strategy,

$$d[N^m, S] = -d[\omega^i N^i + \omega^u N^u, S]_t = \frac{\omega^n}{\omega} (1 - \omega) \Gamma A(t) \beta(t) < 0.$$

Momentum, for the uninformed strategy, is a decreasing function of the conditional variance $H(t)$ and the weight ω^n of mimicking noise traders. It increases over the dividend cycle. The informed strategy is contrarian because the contrarian private information component $N_t^{i,G}$ dominates the overall portfolio behavior. Reversal decreases (i.e., becomes more pronounced) with respect to $H(t)$ and t . The contrarian strategy of the mimicking noise trader is the counterpart of the informed and uninformed strategies.

The uninformed behaves as a trend-chaser because the endogenous market price of risk is positively related to the fundamental. A positive shock to the fundamental induces an increase in the market price of risk, prompting an increase in the uninformed portfolio demand. The behavior of the informed is the opposite. The reason is because the local value of private information, the PIPR, is negatively correlated with the fundamental. The informed acts as a contrarian. Market clearing ensures that the noise trader adopts a contrarian strategy.

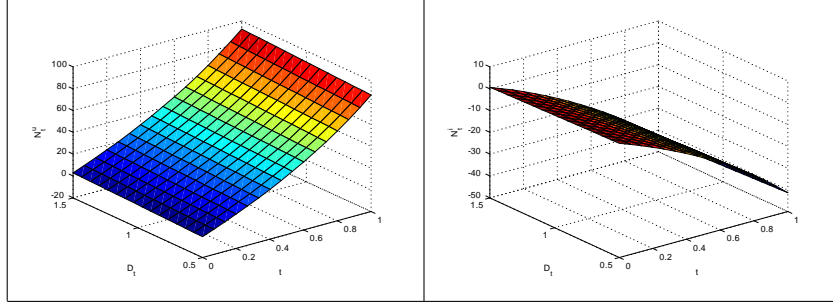


Figure 3-3: Optimal portfolio holdings

Over time, these strategies grow. Contrarians (resp. trend-chasers) become more intense contrarian (resp. trend-chasers). This follows from the fact that fundamental information becomes more important as the dividend date approaches. The market price of risk and the PIPR both become more sensitive to fundamental news over time, prompting investors to amplify their reactions to fundamental news. See Figure 3-3 for illustration. The figure shows the optimal portfolio holdings (z-axis) of the public and the informed investor in the competitive equilibrium as a function of time t (x-axis) and dividends D_t (y-axis). Parameter values are $T = 1$, $\Gamma = 1/8$, $\sigma^D = 0.1$, $\mu^D = 0.05$, $D_0 = 1$, $\mu^\phi = E[G]$, $\sigma^\phi = STD[G]$, $\sigma^\zeta = 0.1$, $G = D_0 + \mu^D T$, $\phi = 0.9 \times G$. The weights of the informed and the mimicking noise traders are $\omega^i = \omega^n = \frac{1}{3}$.

Remark 15 *These findings differ from those in Albuquerque and Miao (2013) and Wang (1993). In Wang's model, the uninformed can be a trend chaser or a contrarian. The informed is a contrarian. The uninformed is a momentum trader if the positive covariance associated with fundamental information dominates the covariance related to the endogenous signal. In Albuquerque and Miao, the uninformed (resp. informed) is a contrarian (resp. trend chaser). This pattern is attributable to the agents' information structures and the properties of the private investment opportunity available to the informed. In this model, the informed invests in the private opportunity and hedges the associated exposure to risk with the stock. The hedging component of the stock demand creates the condition for trend chasing behavior.*

Chapter 4

Asset Pricing with Actively Managed Fund

This chapter develops an equilibrium model with an actively managed fund. Section 4.1 presents the structure of the model. Section 4.2 describes the equilibrium structure and Section 4.3 discusses the properties of equilibrium.

4.1 A General Equilibrium Model

4.1.1 Agents, Assets and Private Information

The economy is populated by *three types of agents*: uninformed investors with heterogeneous risk tolerances, a (representative) informed investor and a (representative) noise trader.

The financial market consists of *two assets* : a riskfree asset which pays interest rate r and a risky asset which pays a liquidating dividend D_T at terminal date T ; the dividend process follows a Brownian Motion with constant drift μ_D and volatility σ_D

$$dD_t = \mu_D dt + \sigma_D dW_{Dt}, \quad t \in [0, T]$$

where W_D is a standard Brownian Motion process with filtration $\mathcal{F}_{(\cdot)}^D$. The process D can be viewed as a fundamental factor that drives the terminal dividend. This risky asset can be regarded as the passive index because there is one stock in the economy.

The informed investor has skill to extract a noisy signal about the terminal div-

idend payment which is $G = D_T + \zeta$, where the signal noise ζ is $\mathcal{N}(0, \sigma_\zeta^2)$. Skill is measured by the precision $v_\zeta = (\sigma_\zeta^2)^{-1}$ of the signal. A high value of σ_ζ^2 corresponds to a low signal precision, and thus skill decreases.

The noise trader who, as in the previous model, has no private information about the asset payoff tries to mimic the behavior of the informed investor but using the false signal ϕ , which is pure noise $\phi \sim \mathcal{N}(0, \sigma_\phi^2)$.

The uninformed investors are heterogeneous in risk tolerances. The risk tolerance specific to each investor is privately known by him. As in Detemple (2002)¹, the uninformed investors' preferences are unknown to the fund manager. These investors are indexed by their risk tolerances. Their distribution is represented by the function $F(\Gamma)$ such that $F(\underline{\Gamma}^u) = 0$ and $F(\bar{\Gamma}^u) = 1$, and the corresponding density is $f(\Gamma)$. These investors observe prices and other quantities that are in the public information filtration, defined by \mathcal{F}_t^m — this public information is endogenously determined in equilibrium

4.1.2 Asset Price and Information Sets

Given the information generated by the dividend process and both the true and false signals (i.e. G and ϕ), trading takes place in continuous time and the stock index S is endogenously determined. Because there are two sources of uncertainty, W_D generated by the fundamental information and ϕ by the noise trader, standard representation theorems can be applied to write any candidate price process as

$$dS_t = \mu_t dt + \sigma_t dW_t = \sigma_t (\theta_t dt + dW_t), \quad S_T = D_T$$

where W_t is the Brownian Motion under the public filtration \mathcal{F}_t^m . As in the previous model, the information filtration \mathcal{F}_t^m is endogenously determined as a result of the

¹Detemple, Jerome B., (2002), Asset pricing in an intertemporal partially-revealing rational expectations equilibrium, Journal of Mathematical Economics, 38, issue 1-2, p. 219-248.

interaction between the fundamental information D_t , the true signal G and the false one ϕ . The difference is that in this model, the dissemination of information is endogenously determined. The coefficients (μ_t, σ_t) of the price process are also endogenous and adapted to the public information \mathcal{F}_t^m . The information filtration $\mathcal{F}_{(\cdot)}^S$ is in the public information $\mathcal{F}_{(\cdot)}^m$. That is, $\mathcal{F}_{(\cdot)}^S \subseteq \mathcal{F}_{(\cdot)}^m$.

As the informed fund manager has private information, his information filtration \mathcal{F}_t^G is enlarged by the signal G and consequently it is equivalent to the sigma algebra $\mathcal{F}_t^G \equiv \{G \vee \mathcal{F}_t^m\}$. Thus, W_t is no longer the standard Brownian Motion with respect to the enlarged filtration \mathcal{F}_t^G . The new standard Brownian Motion W_t^G with respect to \mathcal{F}_t^G is the translated process

$$dW_t^G = dW_t - \theta_t^{G|m}(G) dt$$

where $\sigma_t \theta_t^{G|m}(G)$ is the stock index premium inferred from the private information. Therefore, the stock index from the fund manager's perspective follows the dynamics

$$dS_t = \mu_t dt + \sigma_t \left(dW_t^G + \theta_t^{G|m} dt \right) = \sigma_t \left(\left(\theta_t + \theta_t^{G|m}(G) \right) dt + dW_t^G \right).$$

This representation of the stock index emphasizes how the information advantage is translated to the stock index premium with respect to the private information filtration.

4.1.3 Agent Preferences and Optimal Portfolio Choices

The representative informed investor chooses an optimal portfolio to maximize his utility derived from his terminal wealth minus the disutility incurred by the local

fluctuations of his portfolio's wealth.

$$U^i = \max_{N_t^i} E \left[\underbrace{X_T^i}_{\text{utility from terminal wealth}} - \underbrace{\frac{1}{2\Gamma^i} \int_0^T d\langle X^i \rangle_t}_{\text{disutility from managing the portfolio}} \middle| \mathcal{F}_0^G \right]$$

such that

$$dX_t^i = N_t^i \sigma_t \left(\left(\theta_t + \theta_t^{G|m} (G) \right) dt + dW_t^G \right).$$

Because the anticipative information which is only available to the informed agent is valuable, there is a demand for it from the uninformed investors. Hence, a market for the active fund emerges. The informed agent offers the uninformed ones an opportunity to invest in his actively managed fund in exchange for a fee. This fee is equal to a fraction λ of the terminal wealth of the active portfolio. The fund manager (i.e. the informed agent) chooses the investment management fee λ to maximize his total utility obtained from managing this active fund. To avoid the conflict of interests, the fund manager should have two separate accounts: his own account and the active fund account, and he cannot withdraw/inject money from one account to the other.

The uninformed investors have two investment options - they can invest in the active fund or self-manage their own wealth. If an investor u with risk tolerance Γ^u delegates his wealth to the fund manager, his utility will be

$$\begin{aligned} U^{u,i}(\Gamma^u) &= \underbrace{\mathbb{E} \left[X_T^{u,i} - \frac{1}{2\Gamma^u} \int_0^T d\langle X^{u,i} \rangle_t \middle| \mathcal{F}_0^m \right]}_{\text{controlled by the hidden action of the manager}} + \underbrace{\frac{1}{b_u} \mathbb{E} \left[\int_0^T d\langle X^{u,i} \rangle_t \middle| \mathcal{F}_0^m \right]}_{\text{convenience benefit}} \\ &\quad - \underbrace{\mathbb{E} \left[\lambda (X_T^{u,i} - x_0^{u,i}) \middle| \mathcal{F}_0^m \right]}_{\text{investment fee}} \\ &= \mathbb{E} \left[(1 - \lambda) X_T^{u,i} + \lambda x_0^{u,i} - \left(\frac{1}{2\Gamma^u} - \frac{1}{b_u} \right) \int_0^T d\langle X^{u,i} \rangle_t \middle| \mathcal{F}_0^m \right] \end{aligned}$$

After delegating all wealth to the fund manager, the investor need not worry about the local fluctuations of his wealth dynamics during the whole investment horizon, so I assume that the convenience benefit matches the disutility incurred by the local wealth fluctuations, that is $b_u = 2\Gamma^u$. The investor, therefore, only cares about the expected return at the end of the investment horizon.

Alternatively, the investor can manage his wealth. He chooses an optimal portfolio to maximize his utility

$$U^u(\Gamma^u) = \max_{N_t^u} E \left[X_T^u - \frac{1}{2\Gamma^u} \int_0^T d\langle X^u \rangle_t \middle| \mathcal{F}_0^m \right]$$

subject to

$$dX_t^u = N_t^u \sigma_t (\theta_t dt + dW_t).$$

The fund manager, upon receiving the compensation $\lambda(X_T^u - x_0^u)$ from the investor u , makes an investment on his client's behalf. He maximizes the utility derived from the compensation minus the disutility incurred by the local fluctuations of the active portfolio's wealth.

$$U^{i,i}(\Gamma^u) = \max_{N_t^{u,i}} \mathbb{E}^i \left[\underbrace{\lambda(X_T^{u,i} - x_0^u)}_{\text{utility from compensation}} - \frac{1}{2q} \underbrace{\int_0^T d\langle X^{u,i} \rangle_t}_{\text{disutility from managing the portfolio}} \middle| \mathcal{F}_0^G \right]$$

where q is the fund manager's risk taking level.

Let $\Omega_{\tilde{\lambda}}$ be the set of uninformed investors choosing to invest in the active fund if the investment fee is $\tilde{\lambda}$. The optimal fee λ chosen by the fund manager maximizes his total utility obtained from managing the active fund

$$\lambda = \arg \max_{\tilde{\lambda}} \int_{\Gamma^u \in \Omega_{\tilde{\lambda}}} U^{i,i}(\Gamma^u) f(\Gamma^u) d\Gamma^u.$$

The set Ω_λ is determined endogenously in equilibrium.

Proposition 16 *The optimal number of shares held in the fund manager's account, the self-managed accounts and the active fund accounts are*

$$\begin{aligned} N_t^i &= \Gamma^i \frac{\theta_t + \theta_t^{G|m}(G)}{\sigma_t} \\ N_t^u(\Gamma^u) &= \Gamma^u \frac{\theta_t}{\sigma_t} \text{ if } \Gamma^u \notin \Omega; \quad N_t^{u,i}(\Gamma^u) = \frac{\lambda q \left(\theta_t + \theta_t^{G|m}(G) \right)}{\sigma_t} \text{ if } \Gamma^u \in \Omega \end{aligned}$$

for $t \in [0, T]$ where θ_t is the market price of risk, Ω is the set of the investors in the active fund. The total share holdings in the active fund account are

$$N_t^{active} = \int_{\Gamma^u \in \Omega} \frac{\lambda q \left(\theta_t + \theta_t^{G|m}(G) \right)}{\sigma_t} f(\Gamma^u) d\Gamma^u$$

and the total share holdings by the remaining public investors are

$$N_t^{public} = \int_{\Gamma^u \notin \Omega} \Gamma^u \frac{\theta_t}{\sigma_t} f(\Gamma^u) d\Gamma^u.$$

The optimal demands are of mean-variance structure. The total share holdings in the active fund account increase in the fund manager's risk taking level q . The higher the risk-taking level q , the riskier the active portfolio. As the fund manager cannot observe the specific characteristics of each of his clients, the investment strategy he uses for each of his clients is the same and equal to $N_t^{u,i}$. Thus, the total share holdings in the active fund N_t^{active} are proportional to the number of investors in the fund. On the other hand, the investors who self-manage their portfolios take into account their specific risk tolerances and can customize their portfolios accordingly. As a result, the total share holdings by the remaining public investors N_t^{public} depends on their collective risk tolerances.

Moreover, the investors in the active fund are effectively converted to informed ones with risk tolerances equal to the product of the investment fee λ and the man-

ager's risk-taking level q . This conversion changes both the price informativeness and the economy's risk tolerance. The price informativeness directly affects the investors' decisions as to whether it is worth paying the investment fee to invest in the active fund or not. The economy's risk tolerance directly affects the Sharpe ratio of the risky asset and accordingly the agents' investment decisions. Therefore, each investor's action (either actively manage or self manage his wealth) influences other investors' behaviors.

Remark 17 *The noise trader mimics the informed manager, and the number of shares he demands is*

$$N_t^n = \Gamma^n \frac{\theta_t + \theta_t^{G|m}(\phi)}{\sigma_t}.$$

This demand is based on the false signal ϕ which is completely unrelated to the fundamental. Hence, the quantity $\Gamma^n \frac{\theta_t^{G|m}(\phi)}{\sigma_t}$ represents the speculative demand associated with the pure noise.

4.1.4 Market for Actively Managed Fund

There is an adverse selection in the market for the active fund because only the low risk tolerance investors are attracted to the fund.

Proposition 18 (Adverse Selection and Market Segmentation) *Given the investment management fee λ , there exists a threshold risk tolerance $\Gamma^{u,\lambda} = \mathbb{T}$ such that the investors who have risk tolerances less than or equal to $\Gamma^{u,\lambda}$ will opt to invest in the actively managed fund. The others will self-managed their wealth. The expression for the threshold risk tolerance is*

$$\mathbb{T} = 2(1 - \lambda) \lambda q \mathbb{E} \left[\int_0^T \left(\theta_t + \theta_t^{G|m}(G) \right)^2 \middle| \mathcal{F}_0^m \right] \bigg/ \mathbb{E} \left[\int_0^T \theta_t^2 dt \middle| \mathcal{F}_0^m \right].$$

This proposition shows that the actively managed fund attracts public investors regardless of the fund's performance. Although the investors care about the expected returns, the ones who self-manage their portfolios have additionally disutilities incurred by the local wealth fluctuations throughout the horizon. These disutilities

are attributed to the time spent on conducting fundamental analysis and rebalancing portfolios in response to changes in wealth. On the other hand, the ones who delegate their wealth do not have these disutilities. Due to the heterogeneity in the risk tolerances, the uninformed investors' valuations of the investment service are different. In fact, the investor with the lowest risk tolerance values the service the most, and the one with the highest risk tolerance values the service the least. The adverse selection naturally arises because the fund manager is unable to observe the investors' characteristics. Charging a uniform fee, the fund manager ends up attracting only the low risk tolerance investors to his fund. The adverse selection splits the uninformed population into two separate groups, the low risk tolerance investors who will invest in the active fund and the high risk tolerance ones who will manage their own portfolios. Actually, the low risk tolerance investors are strictly better-off while the fund manager's skill is undervalued.

Remark 19 (Herding behavior) *The investors' herding behavior emerges because of the segmentation in the active fund. Whenever an investor switches to the active fund, he effectively becomes an informed one. This action of his increases the public risk tolerance (i.e the average risk tolerance of the remaining uninformed investors) and also enhances the price informativeness. That the public has higher average risk tolerance means the market can absorb more risk, and thus the risky asset's Sharpe ratio decreases. The lower Sharpe ratio draws more investors to the active fund because the returns on their self-managed portfolios are lower. The inflow of investors continues until prices are sufficiently informative, i.e., enough investors are converted to informed ones. Likewise, when one investor withdraws from the active fund, the Sharpe ratio will be higher and prices are less informative. Thus, more investors start leaving the active fund because they can earn higher returns by self-managing their wealth. The outflow of investors will stop when prices are no longer informative enough. In short, the flow is in the direction such that the marginal benefit of investing in the active fund is equal to the marginal cost of forgoing the public signal.*

Moreover, the market segmentation result simplifies the manager's problem of optimizing his service fee λ . The maximization problem becomes

$$\lambda = \arg \max_{\tilde{\lambda}} \int_{\underline{\Gamma}^u}^{\Gamma^{u,\tilde{\lambda}}} U^{i,i}(\Gamma^u) f(\Gamma^u) d\Gamma^u.$$

The next proposition pins down the optimal investment fee chosen by the manager.

Proposition 20 (Optimal Investment Fee) *The optimal fee λ^* and the corresponding threshold risk tolerance Γ^{u,λ^*} solve the system of equations*

$$\lambda = 1 - \frac{1}{2 \frac{F(\Gamma^{u,\lambda})}{\Gamma^{u,\lambda} f(\Gamma^{u,\lambda})} + 1} \quad (4.1)$$

$$\Gamma^{u,\lambda} = 2(1 - \lambda) \lambda q \mathbb{E} \left[\int_0^T \left(\theta_t + \theta_t^{G|m}(G) \right)^2 \middle| \mathcal{F}_0^m \right] \bigg/ \mathbb{E} \left[\int_0^T \theta_t^2 dt \middle| \mathcal{F}_0^m \right] \quad (4.2)$$

and λ^* must satisfy

$$\frac{1}{2} < \lambda^* < \arg \max_{x \in [\underline{\Gamma}^u, \bar{\Gamma}^u]} \left(1 - \frac{1}{2 \frac{F(x)}{\Gamma^{u,\lambda} f(x)} + 1} \right). \quad (4.3)$$

The equation (4.2) is the demand equation, i.e. the public's response to the fund manager's investment management fee λ . The equation (4.1) is the fund manager's optimal fee given the public population's response described in the equation (4.2). From the equation (4.1), the optimal investment fee is always greater than 1/2; and thus the equation (4.2) implies that the immediate response of the public to an increase in fee is that the investors with risk tolerances equal or close to the risk tolerance threshold $\Gamma^{u,\lambda}$ will switch to self-managing their wealth. Namely, the short-run demand for the active fund decreases when the fee temporarily increases. However, there is no supply curve in the market for the active fund, the fee is set purely by the distribution of risk tolerances of the investors and the manager's risk-taking level, taking the risky asset prices as given. The equation (4.3) guarantees

that the threshold risk tolerance lies in its domain $[\underline{\Gamma}^u, \bar{\Gamma}^u]$

Corollary 21 (Existence of an Equilibrium in Active Fund Market) *For any distribution F , risk tolerance band $[\underline{\Gamma}^u, \bar{\Gamma}^u]$ and the signal G , there exists a bounded interval $[q_{\min}, q_{\max}]$ such that for any $q \in [q_{\min}, q_{\max}]$, there exists a pair $(\lambda, \Gamma^{u,\lambda})$ that solves the fixed point system of equations (4.1) and (4.2).*

In this partial equilibrium analysis (i.e. the asset price process is taken as given), due to the adverse selection in the active fund market, the public investors have an automatic mechanism to protect themselves from the manager's excessive risk-taking behavior. Indeed, the distribution of risk tolerances among the public investors curbs the fund manager's risk-taking levels. The manager cannot execute a strategy that is excessively risky ($q > q_{\max}$) or very conservative ($q < q_{\min}$). Otherwise, no one would want to invest in the active fund. In fact when the range of risk tolerances of the public is bounded, the fund manager following an extremely conservative strategy gives the investors lower expected returns than they could have earned by self-managing their wealth, then the active fund collapses. Likewise, taking excessive risk could increase fund size but up to a certain limit. One reason is that when the active fund attracts enough investors, the fee is fixed by the aggregate investors' valuations of the wealth management service. Thus taking on more risk is undesirable.

4.1.5 Competitive Noisy Rational Expectations Equilibrium

A rational expectations equilibrium (REE) for the economy under consideration is the investment management fee λ , the triple of demands $(N^u(\Gamma^u), N^i, N^{u,i}(\Gamma^u))$ and a price process $dS_t = \mu_t dt + \sigma_t dW_t$, $S_T = D_T$ such that

(i) Individual rationality: N^j the optimal demand for agent $j \in \{u, ui, i\}$, taking the price process as given, and

(ii) Market Clearing:

$$\omega^u \left(\int_{\Gamma^{u,\lambda}}^{\bar{\Gamma}^u} N^u(\Gamma) f(\Gamma) d\Gamma + \int_{\underline{\Gamma}^u}^{\Gamma^{u,\lambda}} N^{u,i} f(\Gamma) d\Gamma \right) + \omega^i N^i + \omega^n N^n = 1.$$

The REE is said to be noisy (NREE) if the informed and uninformed agents' filtrations differ, $\mathcal{F}_{(\cdot)}^u \subset \mathcal{F}_{(\cdot)}^i$.

4.2 The Noisy Rational Expectations Equilibrium Structure

Let $\mathcal{E}(\lambda, \Gamma)$ denote an economy in which an investment fee is λ and a risk tolerance threshold is Γ . Before presenting the NREE in an economy with an endogenous fee λ and risk tolerance threshold Γ , it is useful to define the following quantities

$$\begin{aligned} F(\Gamma) &\equiv \int_{\underline{\Gamma}^u}^{\Gamma} f(\Gamma^u) d\Gamma^u; & E(\Gamma) &\equiv \int_{\Gamma}^{\bar{\Gamma}^u} \Gamma^u f(\Gamma^u) d\Gamma^u \\ w_u^{\mathcal{E}(\lambda, \Gamma)} &\equiv \omega^u E(\Gamma); & w_i^{\mathcal{E}(\lambda, \Gamma)} &\equiv \omega^u \lambda q F(\Gamma) + \omega^i \Gamma^i; & w_n^{\mathcal{E}(\lambda, \Gamma)} &\equiv \omega^n \Gamma^n \\ w^{\mathcal{E}(\lambda, \Gamma)} &\equiv w_i^{\mathcal{E}(\lambda, \Gamma)} + w_n^{\mathcal{E}(\lambda, \Gamma)}; & \mathbb{W}^{\mathcal{E}(\lambda, \Gamma)} &\equiv w_u^{\mathcal{E}(\lambda, \Gamma)} + w_i^{\mathcal{E}(\lambda, \Gamma)} + w_n^{\mathcal{E}(\lambda, \Gamma)} \\ H_t &\equiv \sigma_D^2 (T - t) + \sigma_\zeta^2; & M_t^{\mathcal{E}(\lambda, \Gamma)} &\equiv \left(w_i^{\mathcal{E}(\lambda, \Gamma)} \right)^2 H_t + \left(w_n^{\mathcal{E}(\lambda, \Gamma)} \right)^2 \sigma_\phi^2 \\ \chi^{\mathcal{E}(\lambda, \Gamma)}(t, s) &\equiv \frac{w_i^{\mathcal{E}(\lambda, \Gamma)} \sigma_D^2 (s - t)}{M_t^{\mathcal{E}(\lambda, \Gamma)}} \\ Z^{\mathcal{E}(\lambda, \Gamma)} &\equiv w_i^{\mathcal{E}(\lambda, \Gamma)} G + w_n^{\mathcal{E}(\lambda, \Gamma)} \phi \\ A_t^{\mathcal{E}(\lambda, \Gamma)} &\equiv \left(\frac{H_T}{H_t} \right)^{\frac{w^{\mathcal{E}(\lambda, \Gamma)}}{\mathbb{W}^{\mathcal{E}(\lambda, \Gamma)}}} \left(\frac{M_T^{\mathcal{E}(\lambda, \Gamma)}}{M_t^{\mathcal{E}(\lambda, \Gamma)}} \right)^{1 - \frac{w^{\mathcal{E}(\lambda, \Gamma)}}{\mathbb{W}^{\mathcal{E}(\lambda, \Gamma)}}} \\ \alpha_t^{\mathcal{E}(\lambda, \Gamma)} &\equiv \sigma_D \frac{M_t^{\mathcal{E}(\lambda, \Gamma)} - w_i^{\mathcal{E}(\lambda, \Gamma)} H_t w^{\mathcal{E}(\lambda, \Gamma)}}{H_t M_t^{\mathcal{E}(\lambda, \Gamma)}} \end{aligned}$$

$$\beta_t^{\mathcal{E}(\lambda, \Gamma)} \equiv \sigma_D \left(-M_t^{\mathcal{E}(\lambda, \Gamma)} + \left(w_i^{\mathcal{E}(\lambda, \Gamma)} \right)^2 H_t \right) \frac{w^{\mathcal{E}(\lambda, \Gamma)}}{H_t M_t^{\mathcal{E}(\lambda, \Gamma)}}$$

$$\gamma_t^{\mathcal{E}(\lambda, \Gamma)} \equiv \sigma_D \mu_D \left(-M_t^{\mathcal{E}(\lambda, \Gamma)} + \left(w_i^{\mathcal{E}(\lambda, \Gamma)} \right)^2 H_t \right) w^{\mathcal{E}(\lambda, \Gamma)} \frac{T - t}{H_t M_t^{\mathcal{E}(\lambda, \Gamma)}}.$$

The next proposition identify the necessary condition for the existence of the Noisy Rational Expectations Equilibrium.

Proposition 22 *If a Noisy Rational Expectations Equilibrium exists, there must exist a pair (q_{\min}, q_{\max}) such that the manager's risk-taking level q satisfies $q \in [q_{\min}, q_{\max}]$.*

Similar to partial equilibrium analysis, in general equilibrium, the asymmetric information creates an automatic mechanism for the public investors to protect themselves from the manager's excessive risk-taking. In fact, the distribution of risk tolerances among the public investors coupled with the adverse selection curb the fund manager's risk-taking behaviors. The difference with the partial equilibrium analysis is that there are reactions of the stock market in response to the manager's risk-taking behavior. As before, the manager cannot implement a strategy that is excessively risky or very conservative. Otherwise, no one would want to invest in the active fund. In particular, the distribution of risk tolerances sets the investors' desired returns relative to the market return. On the other hand, the adverse selection induces a lower market risk premium and lower volatility if the fund manager takes on more risk. If the fund manager follows a very conservative strategy gives the investors a lower expected return than they could earn from the market return by self-managing their wealth, then the active fund collapses. By taking on more risk, the manager could only increase fund size up to a certain extent. The reason is that when the active fund attracts enough investors, the market reacts to his excessive risk-taking by substantially lowering the volatility and risk premium, so taking on more risk just reduces the Sharpe ratio of the asset and makes the asset less attractive to hold; this

action is suboptimal for the fund manager.

Define $\theta_t^{\mathcal{E}(\lambda, \Gamma)}$, $\sigma_t^{\mathcal{E}(\lambda, \Gamma)}$, and $\theta_t^{G|m; \mathcal{E}(\lambda, \Gamma)}$ as

$$\theta_t^{\mathcal{E}(\lambda, \Gamma)} \equiv \frac{A_t^{\mathcal{E}(\lambda, \Gamma)} \sigma_D - \left(\alpha_t^{\mathcal{E}(\lambda, \Gamma)} Z^{\mathcal{E}(\lambda, \Gamma)} + \beta_t^{\mathcal{E}(\lambda, \Gamma)} D_t + \gamma_t^{\mathcal{E}(\lambda, \Gamma)} \right)}{\mathbb{W}^{\mathcal{E}(\lambda, \Gamma)}}; \quad \sigma_t^{\mathcal{E}(\lambda, \Gamma)} \equiv A_t^{\mathcal{E}(\lambda, \Gamma)} \sigma_D \quad (4.4)$$

$$\theta_t^{G|m; \mathcal{E}(\lambda, \Gamma)}(G) \equiv \frac{\sigma_D}{H(t)} (G - D_t - \mu_D (T - t)) \quad (4.5)$$

$$- \frac{w_i^{\mathcal{E}(\lambda, \Gamma)} \sigma_D}{M^{\mathcal{E}(\lambda, \Gamma)}(t)} \left(Z^{\mathcal{E}(\lambda, \Gamma)} - w_i^{\mathcal{E}(\lambda, \Gamma)} (D_t + \mu_D (T - t)) \right). \quad (4.6)$$

Proposition 23 (Noisy Rational Expectations Equilibrium) *If there exists a pair $(\lambda^*, \Gamma^{u, \lambda^*})$ that solves the system of fixed point equations (4.1) and (4.2) and λ^* satisfies the condition (4.3), where θ_t and $\theta_t^{G|m}(G)$ are defined by (4.4) and (4.5), then this pair $(\lambda^*, \Gamma^{u, \lambda^*})$ will constitute an NREE equilibrium where the market price of risk, volatility, PIPR are $\theta_t^{\mathcal{E}(\lambda^*, \Gamma^{u, \lambda^*})}$, $\sigma_t^{\mathcal{E}(\lambda^*, \Gamma^{u, \lambda^*})}$ and $\theta_t^{G|m; \mathcal{E}(\lambda^*, \Gamma^{u, \lambda^*})}(G)$. The endogenously generated public signal is*

$$Z^{\mathcal{E}(\lambda^*, \Gamma^{u, \lambda^*})} \equiv w_i^{\mathcal{E}(\lambda^*, \Gamma^{u, \lambda^*})} G + w_n^{\mathcal{E}(\lambda^*, \Gamma^{u, \lambda^*})} \phi$$

The equilibrium stock price is

$$S_t^{\mathcal{E}(\lambda^*, \Gamma^{u, \lambda^*})} = A_t^{\mathcal{E}(\lambda^*, \Gamma^{u, \lambda^*})} D_t + B_t^{\mathcal{E}(\lambda^*, \Gamma^{u, \lambda^*})} Z + F_t^{\mathcal{E}(\lambda^*, \Gamma^{u, \lambda^*})}$$

where

$$B_t^{\mathcal{E}(\lambda^*, \Gamma^{u, \lambda^*})} = \frac{\sigma_D \int_t^T A_s^{\mathcal{E}(\lambda^*, \Gamma^{u, \lambda^*})} \left(\alpha_s^{\mathcal{E}(\lambda^*, \Gamma^{u, \lambda^*})} + \beta_s^{\mathcal{E}(\lambda^*, \Gamma^{u, \lambda^*})} \chi^{\mathcal{E}(\lambda^*, \Gamma^{u, \lambda^*})}(t, s) \right) ds}{\mathbb{W}^{\mathcal{E}(\lambda^*, \Gamma^{u, \lambda^*})}} + \chi^{\mathcal{E}(\lambda^*, \Gamma^{u, \lambda^*})}(t, T)$$

$$\begin{aligned}
F_t^{\mathcal{E}(\lambda^*, \Gamma^u, \lambda^*)} &= A_t^{\mathcal{E}(\lambda^*, \Gamma^u, \lambda^*)} \mu_D (T - t) - \frac{\sigma_D^2}{\mathbb{W}^{\mathcal{E}(\lambda^*, \Gamma^u, \lambda^*)}} \int_t^T \left(A_s^{\mathcal{E}(\lambda^*, \Gamma^u, \lambda^*)} \right)^2 ds \\
&\quad + \frac{\sigma_D}{\mathbb{W}^{\mathcal{E}(\lambda^*, \Gamma^u, \lambda^*)}} \int_t^T A_s^{\mathcal{E}(\lambda^*, \Gamma^u, \lambda^*)} \gamma_s^{\mathcal{E}(\lambda^*, \Gamma^u, \lambda^*)} ds.
\end{aligned}$$

The Brownian Motion in the public information is the process

$$dW_t^{\mathcal{E}(\lambda^*, \Gamma^u, \lambda^*)} = dW_{Dt} - \theta_t^{m|D; \mathcal{E}(\lambda^*, \Gamma^u, \lambda^*)}$$

where

$$\theta_t^{m|D; \mathcal{E}(\lambda^*, \Gamma^u, \lambda^*)} = \frac{w_i^{\mathcal{E}(\lambda^*, \Gamma^u, \lambda^*)} \sigma_D \left(Z - w_i^{\mathcal{E}(\lambda^*, \Gamma^u, \lambda^*)} (D_t + \mu_D (T - t)) \right)}{\left(w_i^{\mathcal{E}(\lambda^*, \Gamma^u, \lambda^*)} \right)^2 H_t + \left(w_n^{\mathcal{E}(\lambda^*, \Gamma^u, \lambda^*)} \right)^2 \sigma_\phi^2}.$$

Miscellaneous information such as the private information G and the purely noisy information ϕ are disseminated to the public in the form of $Z^{\mathcal{E}(\lambda^*, \Gamma^u, \lambda^*)} \equiv w_i^{\mathcal{E}(\lambda^*, \Gamma^u, \lambda^*)} G + w_n^{\mathcal{E}(\lambda^*, \Gamma^u, \lambda^*)} \phi$. This quantity Z is the public signal endogenously generated in the equilibrium. Thus, by observing the total trades of the fund manager and the noise trader, the public investors partially learn about the anticipative information G . However the private informational content (which is measured by the weight $\frac{w_i^{\mathcal{E}(\lambda^*, \Gamma^u, \lambda^*)}}{w_n^{\mathcal{E}(\lambda^*, \Gamma^u, \lambda^*)}}$) collected from the market data is affected by the active fund sector. More precisely, the investors investing in the active fund are effectively converted to informed ones with risk tolerances equal to the product of the investment fee λ and the manager's risk-taking level q . This conversion changes both the price informativeness and the economy's risk tolerance.

First, price informativeness plays a major role in determining the equilibrium risky asset prices. In particular, the fund manager's risk-taking behavior or managerial skill has a direct impact on the number of public investors drawn to the active fund, so

changes in the active fund market affect the private informational content in the public signal. As the public investors depend purely on the fundamental information D_t and the endogenous public signal Z , their trades are affected by the changes in the active fund market. In short, the presence of the active fund sector endogenizes the process of anticipative information dissemination, affecting the aggregate demand of the risky asset, and consequently the equilibrium prices.

Second, changes in the average public risk tolerance are also crucial in determining the equilibrium market price of risk (i.e. Sharpe ratio) and volatility. The market price of risk and volatility reflect the public perception of the uncertainties in the stock market. If the public has collectively high risk tolerance, the market price of risk and volatility must be low. Because of the segmentation in the market for the active fund, the average public risk tolerance will be higher (lower) if more (less) investors invest in the active fund. Therefore, the presence of the active fund influences the public capacity to absorb the market risk, affecting the equilibrium stock price.

Lastly, in contrast to a popular belief that a higher skilled fund manager can attract more investors to his active fund, the chapter shows that it might not be the case. In fact, changes in the managerial skill induce changes in the investment management fee and the threshold risk tolerance. As a result, all three factors - private information precision, price informativeness and economy's overall risk tolerance - change. The three simultaneously interacting effects plus the distribution of risk tolerances among the public investors yield new asset pricing implications which will be described and illustrated numerically in Section 4.3.

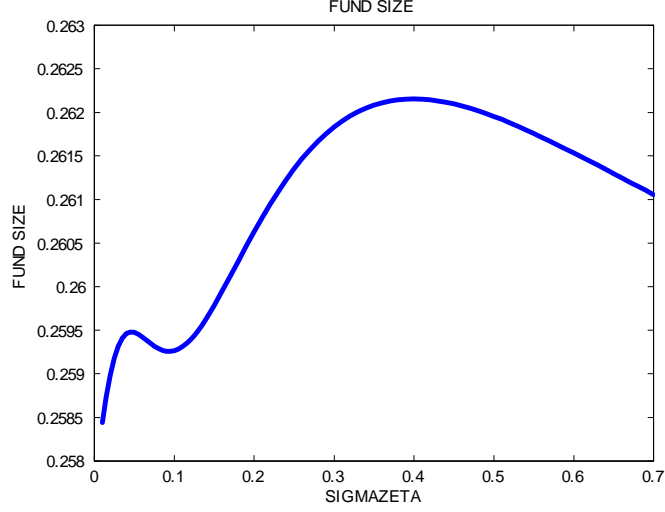


Figure 4.1: The relationship between Managerial Skill and Fund Size is non-monotonic. Parameters $\omega_u = 0.2$, $\omega_i = 0.4$, $\omega_n = 0.4$, $q = 0.8$, $\underline{\Gamma}^u = 0.1$, $\bar{\Gamma}^u = 1$, $\Gamma^i = 2$, $\Gamma^n = 2$, $\sigma_\phi = 0.1$, $\mu_D = 0.1$, $\sigma_D = 0.1$.

4.3 Properties of Equilibrium

This section examines an economy in which the investors' risk tolerances are uniformly distributed

$$f(\Gamma) = \frac{1}{\bar{\Gamma}^u - \underline{\Gamma}^u}; \quad F(\Gamma) = \int_{\underline{\Gamma}^u}^{\Gamma} f(\Gamma) d\Gamma = \frac{\Gamma - \underline{\Gamma}^u}{\bar{\Gamma}^u - \underline{\Gamma}^u}.$$

The corresponding fixed point equations become

$$\begin{aligned} \lambda &= 1 - \frac{1}{2} \frac{1}{2 - \underline{\Gamma}^u / \Gamma^{u,\lambda}} \\ \Gamma^{u,\lambda} &= 2(1 - \lambda) \lambda q \mathbb{E} \left[\int_0^T \left(\theta_t + \theta_t^{G|m}(G) \right)^2 \middle| \mathcal{F}_0^m \right] \bigg/ \mathbb{E} \left[\int_0^T \theta_t^2 dt \middle| \mathcal{F}_0^m \right] \end{aligned}$$

and

$$\frac{1}{2} < \lambda^* < 1 - \frac{1}{2} \frac{1}{2 - \underline{\Gamma}^u / \bar{\Gamma}^u}.$$

4.3.1 Can Fund Size Predict Managerial Skill?

Among thousands of active funds, an investor necessarily needs to identify truly skilled fund managers. The identification problem is far more complicated because as in Figure (4.1), the relationship between fund size and managerial skill is not monotonic. Contrary to popular belief that with higher managerial skill, the firm will expand and increase its scale, in the market for active funds investors should act cautiously because fund size is not a reliable indicator of the manager's skill level. The relationship between the two, in fact, has a double hump shape.

What happens is as follows. First, an increase in managerial skill immediately increases the private information precision and the price informativeness. The investor who knows that the manager has higher skill has two options: he either invests in the active fund or self-manages his wealth. It is tempting to conclude that if the private information effect dominates the price informativeness effect, he should invest in the active fund and vice versa. If this were true, the relationship between fund size and managerial skill would have an inverted U-shape. Particularly, starting at very low skill, an increase in skill significantly increases the private information precision, i.e., the private information precision effect dominates the price informativeness one. When skill is sufficiently high, an increase in skill no longer produces such a strong private information precision effect, and thus the price informativeness effect dominates it. This means that as skill increases, the investors are first drawn to the active fund but then they leave it because they are better off using the public signal rather than paying the investment fee to extract the benefit from the private information. However, the relationship observed in Figure (4.1) is a double hump shaped curve. Actually, the investors not just evaluate the immediate benefits of the private information precision and the price informativeness but also take into account the herding behavior of their peers. This herding behavior is explained in Remark 19.

The investors' herding behavior emerges because of the segmentation in the active fund. Whenever an investor switches to the active fund, there is an inflow of investors to the active fund. Likewise, if one investor withdraws from the active fund, there is an outflow of investors from the active fund. In fact, some investors are herded together, and the flow is in the direction such that the marginal benefit of obtaining the anticipative information is equal to the marginal cost of forgoing the public signal. Therefore, the equilibrium fund size is determined by the collective actions of all agents. In fact, there are three forces simultaneously interacting in response to the manager's higher skill: private information precision, price informativeness and overall risk tolerance of the economy; the overall result is a double hump shaped curve of the fund size - investment fee relationship.

In particular, the private information precision has a positive effect on the equilibrium fund size. The price informativeness, on the contrary, has a negative effect on the fund size because some investors are better off using the endogenous public signal instead of paying the fund manager some fee. Specific to the economy setup in Figure (4.1), the second hump is the result of the risk tolerance effect. In general, the risk tolerance effect is ambiguous. Any investor needs to take into account his peers' herding behaviors. Therefore, the investor essentially compares the two final results in order to decide whether to join the active fund or not, namely whether the benefit of the anticipative information if he joins the active fund is greater than the higher Sharpe ratio he earns if he does otherwise. As the investors' risk tolerances have non-linear effects on the equilibrium risky asset prices, the herding direction is ambiguous.

4.3.2 Investors' Reaction to Fund Manager Risk Taking

The riskier the active portfolio, the more investors invest in the active fund because by delegating their wealth, the investors only care about the expected return on

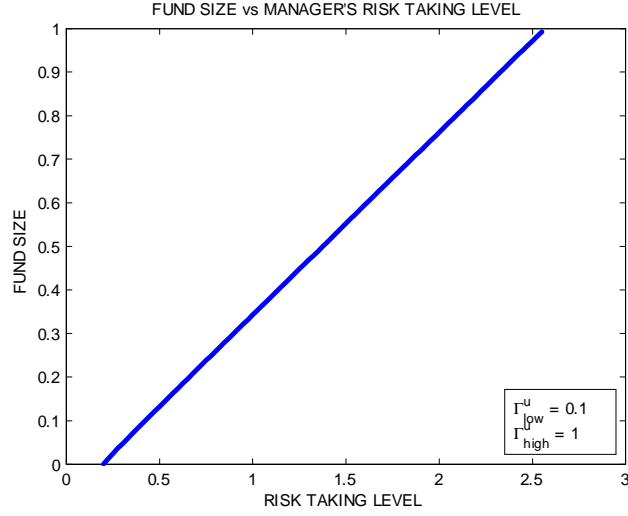


Figure 4.2: Fund Size is increasing in Manager's Risk Taking Level $q \in [q_{\min}, q_{\max}]$. If $q < q_{\min}$ or $q > q_{\max}$, no equilibrium exists. Parameters $\omega_u = 0.2, \omega_i = 0.4, \omega_n = 0.4, \underline{\Gamma}^u = 0.1, \bar{\Gamma}^u = 1, \Gamma^i = 2, \Gamma^n = 2, \sigma_\zeta = 0.1, \sigma_\phi = 0.1, \mu_D = 0.1, \sigma_D = 0.1$

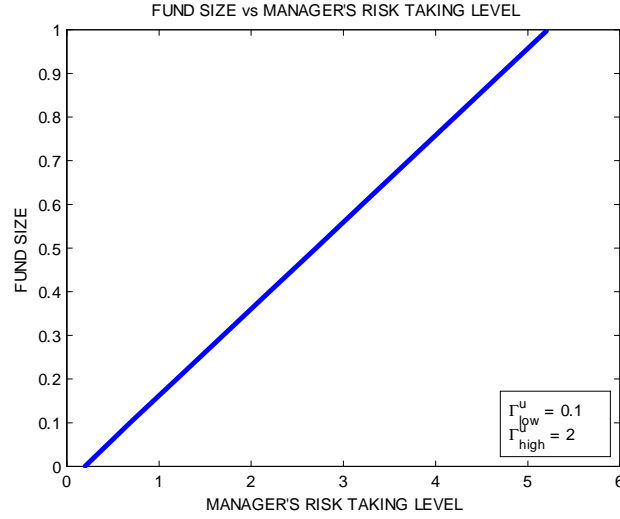


Figure 4.3: Fund Size is increasing in Manager's Risk Taking Level $q \in [q_{\min}, q_{\max}]$. If $q < q_{\min}$ or $q > q_{\max}$, no equilibrium exists. Parameters $\omega_u = 0.2, \omega_i = 0.4, \omega_n = 0.4, \underline{\Gamma}^u = 0.1, \bar{\Gamma}^u = 2, \Gamma^i = 2, \Gamma^n = 2, \sigma_\zeta = 0.1, \sigma_\phi = 0.1, \mu_D = 0.1, \sigma_D = 0.1$

the active portfolio while the fluctuations in the active account are taken care of by the fund manager. The more risk the fund manager takes, the higher the expected return and the larger the number of investors attracted. The fund size, therefore, increases in the manager's risk-taking level. However, if the fund manager takes on more risk, the economy's overall risk tolerance is essentially much higher. Therefore, the risk premium decreases. A lower risk premium makes some of the investors who are self-managing their portfolios switch to the active fund because the returns on their self-managed portfolios are lower. Therefore, investing in the active fund can insure uninformed investors against the competition with the informed and the very high risk tolerance investors.

In different types of economies (i.e. different distribution of risk tolerances among public investors), the manager's possible risk-taking levels are in different ranges. For example, consider the two economies where $\Gamma^i = 2$, and $\Gamma^n = 2$, while the tolerance Γ^u ranges from $\underline{\Gamma}^u = 0.1$ to $\bar{\Gamma}^u = 1$ in Figure (4.2) and $\bar{\Gamma}^u = 2$ in Figure (4.3). In all economies, the fund manager could take risk q which is even higher than the highest risk tolerance of the public investors'. More precisely, in Figure (4.2) the possible range of manager's risk taking level is $[q_{\min} = 0.2, q_{\max} = 2.55]$ while the uninformed public population risk tolerance band is $[\underline{\Gamma}^u = 0.1, \bar{\Gamma}^u = 1]$. In Figure (4.3), the range for manager's risk-taking level so that an equilibrium exists is now $[q_{\min} = 0.2, q_{\max} = 5.2]$ while the public risk tolerance ranges from $[\underline{\Gamma}^u = 0.1, \bar{\Gamma}^u = 2]$. Therefore, the distribution of risk tolerances among investors affects the fund manager's risk-taking behavior.

Moreover, by increasing the risk-taking level, the active fund sector cannot always attract the whole public population as depicted in Figure (4.2) or (4.3). As soon as the distribution of the risk tolerances is polarized, the fund manager fails to attract as many investors as he wants by taking on more risk. Because of polarization in

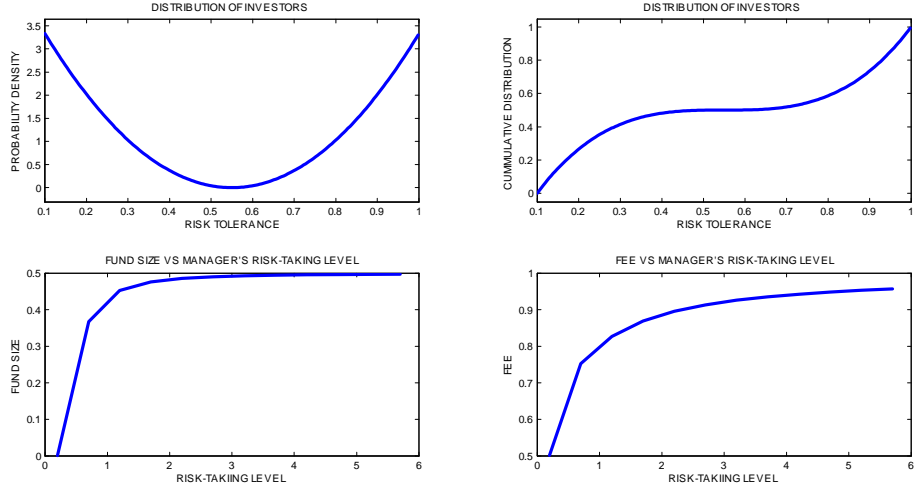


Figure 4.4: Fund Size is increasing in Manager's risk-taking level. Parameters $\omega_u = 0.2$, $\omega_i = 0.4$, $\omega_n = 0.4$, $\underline{\Gamma}^u = 0.1$, $\bar{\Gamma}^u = 1$, $\Gamma^i = 2$, $\Gamma^n = 2$, $\sigma_\zeta = 0.1$, $\sigma_\phi = 0.1$, $\mu_D = 0.1$, $\sigma_D = 0.1$

investors' risk tolerances, the market reacts immediately to his risk-taking behavior. Consider a polarized distribution which is mathematically represented by

$$\begin{aligned}
 f(\Gamma) &= \frac{3(\Gamma - \frac{\underline{\Gamma}^u + \bar{\Gamma}^u}{2})^2}{(\bar{\Gamma}^u - \frac{\underline{\Gamma}^u + \bar{\Gamma}^u}{2})^3 - (\underline{\Gamma}^u - \frac{\underline{\Gamma}^u + \bar{\Gamma}^u}{2})^3}, F(\Gamma) \\
 &= \int_{\underline{\Gamma}^u}^{\Gamma} f(\Gamma) d\Gamma = \frac{(\Gamma - \frac{\underline{\Gamma}^u + \bar{\Gamma}^u}{2})^3 - (\underline{\Gamma}^u - \frac{\underline{\Gamma}^u + \bar{\Gamma}^u}{2})^3}{(\bar{\Gamma}^u - \frac{\underline{\Gamma}^u + \bar{\Gamma}^u}{2})^3 - (\underline{\Gamma}^u - \frac{\underline{\Gamma}^u + \bar{\Gamma}^u}{2})^3}
 \end{aligned}$$

Figure (4.4) plots the probability density (upper left), the cumulative distribution (upper right), the relationship between the fund size and the manager's risk-taking (lower left), and the relationship between the fee and the manager's risk-taking (lower right). The manager can at most attract the low risk tolerance group but is never able to draw any investor in the high risk tolerance group to his fund. The investors with moderate risk tolerances are few, so the benefit of attracting them to the fund is very small while taking more risk incurs considerable costs. Therefore, the polarization limits the expansion of the active fund.

4.3.3 Market Reaction to Manager's Risk Taking

In this section, the market's reactions to the manager's risk-taking level are examined. As the manager's risk-taking level increases, the market price of risk, volatility and risk premium all decrease. Illustrated in Figure (4.5): The market price of risk decreases because more uninformed investors switch to the active fund, only the very high risk tolerance ones self-manage their portfolios. As a result, the public average risk tolerance is higher, and those remaining public investors are able to absorb more market risk. This leads to a decrease in the market price of risk, see Figure (4.7). A decrease in volatility is the consequence of an increase in price informativeness. Because the uninformed investors who are in the active fund are effectively converted to informed ones, the price informativeness increases. As shown in Detemple, Rindisbacher & Truong (2014) the anticipative information stabilizes stock prices, an increase in price informativeness necessarily induces a decrease in the equilibrium volatility. Because both the market price of risk and volatility decrease in response to an increase in the manager's risk taking level, the risk premium which is a product of market price of risk and volatility decreases as shown in Figure (4.6).

Therefore, the market reaction to the manager's increasing risk-taking behavior is to reduce the equilibrium market price of risk, volatility and risk premium.

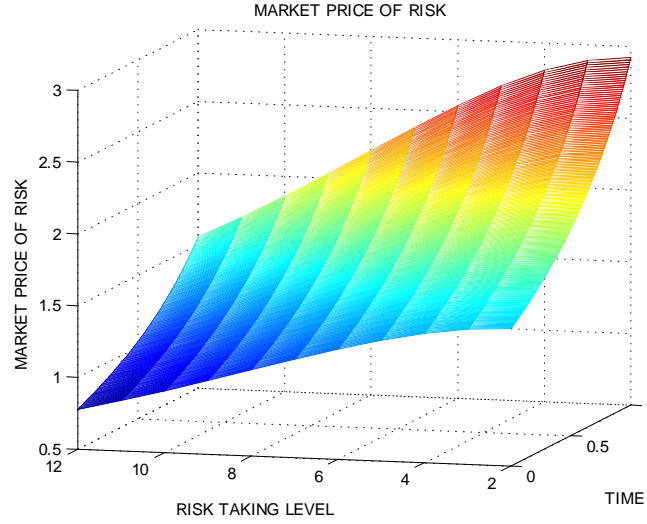


Figure 4-5: Market price of risk decreases as manager's risk taking level increases. Parameters: $\omega_u = 0.6$, $\omega_i = 0.2$, $\omega_n = 0.2$, $\underline{\Gamma}^u = 1$, $\bar{\Gamma}^u = 5$, $\Gamma^i = 5$, $\Gamma^n = 5$, $\sigma_\zeta = 0.1$, $\sigma_\phi = 0.1$, $\mu_D = 0.1$, $\sigma_D = 0.1$, $q \in [2, 12]$

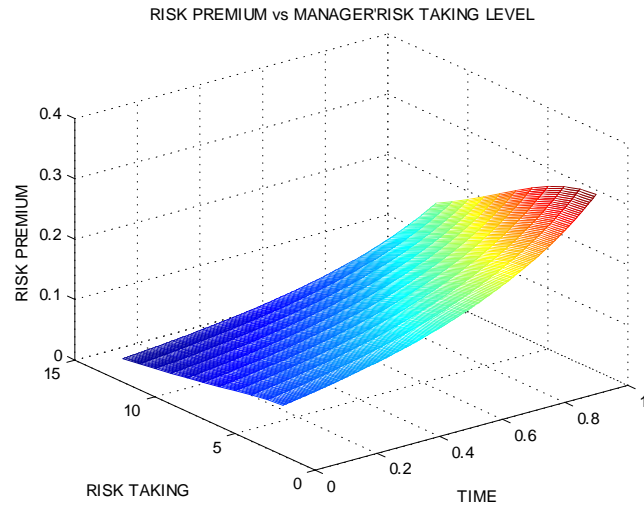


Figure 4-6: Risk premium decreases as manager's risk taking level increases. Parameters: $\omega_u = 0.6$, $\omega_i = 0.2$, $\omega_n = 0.2$, $\underline{\Gamma}^u = 1$, $\bar{\Gamma}^u = 5$, $\Gamma^i = 5$, $\Gamma^n = 5$, $\sigma_\zeta = 0.1$, $\sigma_\phi = 0.1$, $\mu_D = 0.1$, $\sigma_D = 0.1$, $q \in [2, 12]$

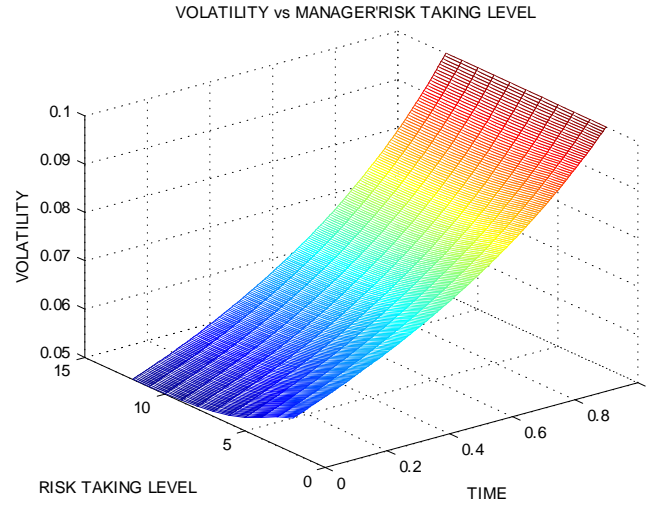


Figure 4.7: Volatility decreases as manager's risk taking level increases. Parameters: $\omega_u = 0.6$, $\omega_i = 0.2$, $\omega_n = 0.2$, $\underline{\Gamma}^u = 1$, $\bar{\Gamma}^u = 5$, $\Gamma^i = 5$, $\Gamma^n = 5$, $\sigma_\zeta = 0.1$, $\sigma_\phi = 0.1$, $\mu_D = 0.1$, $\sigma_D = 0.1$, $q \in [2, 12]$

Chapter 5

Conclusion

This thesis examines the structure and properties of non-stationary noisy rational expectations equilibria in models with continuous trading and discrete dividend payment dates. The first model studies an economy populated by an informed investor, an uninformed investor and a noise trader. The second model considers the case where the informed agent tries to sell his private information in the form of actively managed funds in exchange for a fee.

In the first model, equilibrium prices fail to be weak-form efficient. Public information is carried by the price-fundamental pair. Informed trading has a stabilizing effect, as it reduces the volatility of the stock price. Over the dividend cycle, the stock price volatility, the price of risk and the covariance between the stock price and the price of risk all increase.

The second model provides a micro-foundation for the existence and asset pricing implications of the actively managed fund industry. The endogenization of private information dissemination has a direct impact on the actively managed fund sector. First, the active fund exists to insure the low risk tolerance investors against the competition with the informed agents and the high risk tolerance investors in the financial market. Investment is not all about the returns but also the time and effort spent on managing a portfolio, so the low risk tolerance investors are strictly better off with this alternative investment. Those should invest in the active funds. Second, the active fund sector is distinct from other industries to the extent that the size

of the actively managed fund is not monotonically increasing in the fund manager's skill level. Therefore, fund size is not a reliable indicator of the managerial skill. The market also reacts to the manager's increasing risk-taking behavior by reducing the market price of risk, volatility and risk premium. Last, the public investors have an automatic mechanism to protect themselves against the manager's excessive risk-taking. Indeed, the distribution of risk tolerances among the public investors together with the adverse selection problem in the active fund market set limits on the fund manager's risk-taking levels. The manager cannot follow a strategy that is excessively risky or very conservative, otherwise no one would want to invest in the active fund.

Appendix

Proof of Proposition 4. The aggregate demand function $N_t^a \equiv \omega^u N_t^u + \omega^i N_t^i + \omega^n N_t^n$ is,

$$N_t^a = \omega^u \Gamma \frac{\sigma_t^S \theta_t^m}{(\sigma_t^S)^2} + \omega^i \Gamma \frac{\sigma_t^S \left(\theta_t^m + \theta_t^{G|m}(G) \right)}{(\sigma_t^S)^2} + \omega^n \Gamma \frac{\sigma_t^S \left(\theta_t^m + \theta_t^{G|m}(\phi) \right)}{(\sigma_t^S)^2}$$

where the function $\theta_t^{G|m}(x)$ is endogenous. Conjecture that $\theta_t^{G|m}(x)$ is an affine function of x and let $\Theta_t(z; \omega) \equiv \omega^i \theta_t^{G|m}(x_1) + \omega^n \theta_t^{G|m}(x_2)$ where $z = \omega^i x_1 + \omega^n x_2$ and $\omega = \omega^i + \omega^n$. Under this conjecture the aggregate demand function becomes $N_t^a = \Gamma (\theta_t^m + \Theta_t(Z; \omega)) / \sigma_t^S$ and, at equilibrium, $N_t^a = 1$, $\sigma_t^S \theta_t^m = (\sigma_t^S)^2 / \Gamma - \sigma_t^S \Theta_t(Z; \omega)$. Information revealed in equilibrium includes the noisy translation of the private signal $Z = \omega^i G + \omega^n \phi$. Thus, $F_{(\cdot)}^m \supseteq F_{(\cdot)}^{D,Z}$. Suppose that $F_{(\cdot)}^m = F_{(\cdot)}^{D,Z}$. Given that,

$$G = D_T + \zeta = D_t + \mu^D (T - t) + \int_t^T \sigma^D dW_s^D + \zeta$$

$$Z = \omega^i G + \omega^n \phi = \omega^i (D_t + \mu^D (T - t)) + \omega^i \left(\int_t^T \sigma^D dW_s^D + \zeta \right) + \omega^n \phi$$

the conditional density at time t of the signal is $p_t^G(x) = \frac{1}{\sigma_t^{G|D,Z}} n \left(\frac{x - \mu_t^{G|D,Z}}{\sigma_t^{G|D,Z}} \right)$ where,

$$\mu_t^{G|D,Z} = D_t + \mu^D (T - t) + \kappa_t [Z - \omega^i (D_t + \mu^D (T - t)) - \omega^n \mu^\phi]$$

$$\left(\sigma_t^{G|D,Z} \right)^2 = \left((\sigma^D)^2 (T - t) + (\sigma^\zeta)^2 \right) (1 - \kappa_t \omega^i) \equiv H(t) (1 - \kappa_t \omega^i)$$

$$\kappa_t = \frac{\omega^i \left((\sigma^D)^2 (T-t) + (\sigma^\zeta)^2 \right)}{M(t)} = \frac{\omega^i H(t)}{M(t)}$$

$$M(t) = (\omega^i)^2 \left((\sigma^D)^2 (T-t) + (\sigma^\zeta)^2 \right) + (\omega^n)^2 (\sigma^\phi)^2 = (\omega^i)^2 H(t) + (\omega^n)^2 (\sigma^\phi)^2$$

$(M(t))$ is the variance of $Z - \omega^i (D_t + \mu^D (T-t)) = \omega^i \left(\int_t^T \sigma^D dW_s^D + \zeta \right) + \omega^n \phi$.

Ito's lemma gives the PIPR,

$$\theta_t^{G|m}(x) = \frac{x - \mu_t^{G|D,Z}}{(\sigma_t^{G|D,Z})^2} (1 - \kappa_t \omega^i) \sigma^D = \frac{x - \mu_t^{G|D,Z}}{H(t)} \sigma^D.$$

The PIPR for dividend risk, $\theta_t^{G|m}(x)$, is affine in x , as conjectured.

The information revealed in equilibrium is contained in,

$$\begin{aligned} \Theta_t(Z; \omega) &\equiv \omega^i \theta_t^{G|m}(G) + \omega^n \theta_t^{G|m}(\phi) = \frac{Z - (\omega^i + \omega^n) \mu_t^{G|D,Z}}{H(t)} \sigma^D \equiv \frac{Z - \omega \mu_t^{G|D,Z}}{H(t)} \sigma^D \\ &= \frac{Z - \omega \left((1 - \kappa_t \omega^i) (D_t + \mu^D (T-t)) + \kappa_t (Z - \omega^n \mu^\phi) \right)}{H(t)} \sigma^D \\ &= \left(\frac{1 - \kappa_t \omega}{H(t)} Z - \omega \frac{1 - \kappa_t \omega^i}{H(t)} D_t - \omega \frac{(1 - \kappa_t \omega^i) \mu^D (T-t) - \omega^n \kappa_t \mu^\phi}{H(t)} \right) \sigma^D \\ &\equiv \alpha(t) Z + \beta(t) D_t + \gamma(t) \end{aligned}$$

where $\omega = \omega^i + \omega^n$, and is indeed equivalent to Z provided $\alpha(t) \neq 0$, i.e., $1 - \kappa_t \omega \neq 0$ for t in a neighborhood of 0. At $t = 0$, the condition is equivalent to, $1 - \kappa_0 \omega \neq 0 \iff \omega^n (\sigma^\phi)^2 - \omega^i \left((\sigma^D)^2 T + (\sigma^\zeta)^2 \right) \neq 0$. If the condition fails at $t = 0$, it holds at $t = 0_+$, so Z is immediately revealed in this case as well. Moreover, the pair (Z, D) is a sufficient statistic, in equilibrium, for the PIPR and the conditional density of the signal. This suggests that the pair is a sufficient statistic for the rest of the equilibrium as well. This still needs to be verified.

Suppose that uninformed agents use $F_{(\cdot)}^{D,Z}$ to forecast the future dividend and assess the price of risk θ^m . In equilibrium, $\mu_t^S = \sigma_t^S \theta_t^m = (\sigma_t^S)^2 / \Gamma - \sigma_t^S \Theta_t(Z; \omega)$, which is affine with respect to the pair (Z, D) . The volatility structure remains to

be identified. Assuming volatility coefficients are functions of time and simplifying yields,

$$\begin{aligned}
S_t &= \mathbb{E} \left[D_T - \int_t^T \mu_s^S ds \middle| \mathcal{F}_t^m \right] = D_t + \mu^D(T-t) + \sigma^D \mathbb{E} \left[W_T^D - W_t^D \middle| \mathcal{F}_t^{Z,D} \right] \\
&\quad - \frac{1}{\Gamma} \int_t^T (\sigma_s^S)^2 ds + \int_t^T \mathbb{E} \left[\sigma_s^S (\alpha(s)Z + \beta(s)D_s + \gamma(s)) \middle| \mathcal{F}_t^{Z,D} \right] \\
&= D_t + \mu^D(T-t) - \frac{1}{\Gamma} \int_t^T (\sigma_s^S)^2 ds + \int_t^T \sigma_s^S \gamma(s) ds + \left(\int_t^T \sigma_s^S \alpha(s) ds \right) Z \\
&\quad + \sigma^D \mathbb{E} \left[W_T^D - W_t^D \middle| \mathcal{F}_t^{Z,D} \right] + \int_t^T \sigma_s^S \beta(s) \mathbb{E} \left[D_s \middle| \mathcal{F}_t^{Z,D} \right] ds \\
&\equiv D_t + G_0(t, T) Z + \widehat{F}(\sigma^S, t) + \sigma^D \mathbb{E} \left[W_T^D - W_t^D \middle| \mathcal{F}_t^{Z,D} \right] \\
&\quad + \int_t^T \sigma_s^S \beta(s) \mathbb{E} \left[D_s \middle| \mathcal{F}_t^{Z,D} \right] ds
\end{aligned}$$

where $G_0(t, T) = \int_t^T \sigma_s^S \alpha(s) ds$ and $\widehat{F}(\sigma^S, t) = \mu^D(T-t) - (1/\Gamma) \int_t^T (\sigma_s^S)^2 ds + \int_t^T \sigma_s^S \gamma(s) ds$. Moreover,

$$\begin{aligned}
\mathbb{E} \left[\int_t^T \sigma^D dW_s^D \middle| \mathcal{F}_t^{Z,D} \right] &= \lambda(t, T) (Z - \omega^i (D_t + \mu^D(T-t)) - \omega^n \mu^\phi) \\
&= \lambda(t, T) Z - \omega^i \lambda(t, T) D_t - \lambda(t, T) (\omega^i \mu^D(T-t) + \omega^n \mu^\phi)
\end{aligned}$$

$$\begin{aligned}
\mathbb{E} \left[D_s \middle| \mathcal{F}_t^{Z,D} \right] &= D_t + \mu^D(T-t) + \lambda(t, s) (Z - \omega^i (D_t + \mu^D(T-t)) - \omega^n \mu^\phi) \\
&= (D_t + \mu^D(T-t)) (1 - \omega^i \lambda(t, s)) + \lambda(t, s) Z - \omega^n \lambda(t, s) \mu^\phi
\end{aligned}$$

where $\lambda(t, s) = \frac{\omega^i(\sigma^D)^2(s-t)}{M(t)}$, so that,

$$\int_t^T \sigma_s^S \beta(s) \mathbb{E} \left[D_s | \mathcal{F}_t^{Z,D} \right] ds = G_1(t, T) (D_t + \mu^D (T - t)) + G_2(t, T) (Z - \omega^n \mu^\phi)$$

$$G_1(t, T) = \int_t^T \sigma_s^S \beta(s) (1 - \omega^i \lambda(t, s)) ds, \quad G_2(t, T) = \int_t^T \sigma_s^S \beta(s) \lambda(t, s) ds.$$

Hence,

$$\begin{aligned} S_t &= D_t + G_0(t, T) Z + \widehat{F}(\sigma^S, t) + \lambda(t, T) Z - \omega^i \lambda(t, T) D_t \\ &\quad - \lambda(t, T) (\omega^i \mu^D (T - t) + \omega^n \mu^\phi) + G_1(t, T) (D_t + \mu^D (T - t)) \\ &\quad + G_2(t, T) (Z - \omega^n \mu^\phi) \\ &= (1 - \omega^i \lambda(t, T) + G_1(t, T)) D_t + (G_0(t, T) + \lambda(t, T) + G_2(t, T)) Z + \widehat{F}(\sigma^S, t) \\ &\quad + (-\omega^i \lambda(t, T) + G_1(t, T)) \mu^D (T - t) - (\lambda(t, T) + G_2(t, T)) \omega^n \mu^\phi \\ &\equiv A(t) D_t + B(t) Z + F(t) \end{aligned}$$

where,

$$A(t) = 1 - \omega^i \lambda(t, T) + G_1(t, T) = 1 - \omega^i \lambda(t, T) + \int_t^T \sigma_s^S \beta(s) (1 - \omega^i \lambda(t, s)) ds$$

$$B(t) = G_0(t, T) + \lambda(t, T) + G_2(t, T) = \lambda(t, T) + \int_t^T \sigma_s^S (\alpha(s) + \beta(s) \lambda(t, s)) ds$$

$$F(t) = \widehat{F}(\sigma^S, t) + (A(t) - 1) \mu^D (T - t) - \left(B(t) - \int_t^T \sigma_s^S \alpha(s) ds \right) \omega^n \mu^\phi.$$

An application of Ito's lemma shows that $\sigma_t^S = A(t) \sigma^D$. The volatility coefficient is deterministic as conjectured. This validates the construction of the equilibrium stock

price to this stage. Substituting in the coefficients above gives,

$$A(t) = 1 - \omega^i \lambda(t, T) + \sigma^D \left(\int_t^T A(s) \beta(s) (1 - \omega^i \lambda(t, s)) ds \right)$$

$$B(t) = \lambda(t, T) + \sigma^D \left(\int_t^T A(s) (\alpha(s) + \beta(s) \lambda(t, s)) ds \right)$$

$$F(t) = \widehat{F}(A(t) \sigma^D, t) + (A(t) - 1) \mu^D (T - t) - I(t) \omega^n \mu^\phi$$

$$I(t) = B(t) - \sigma^D \int_t^T A(s) \alpha(s) ds.$$

Inserting $\widehat{F}(A(t) \sigma^D, t) = \mu^D (T - t) - \frac{1}{\Gamma} (\sigma^D)^2 \int_t^T A(s)^2 ds + \sigma^D \int_t^T A(s) \gamma(s) ds$ in the last coefficient and collecting terms leads to,

$$F(t) = A(t) \mu^D (T - t) - \frac{(\sigma^D)^2}{\Gamma} \int_t^T A(s)^2 ds + \sigma^D \int_t^T A(s) \gamma(s) ds - I(t) \omega^n \mu^\phi$$

$$I(t) = \lambda(t, T) + \sigma^D \int_t^T A(s) \beta(s) \lambda(t, s) ds.$$

In these expressions, with $\omega = \omega^i + \omega^n$,

$$\alpha(t) = \frac{1 - \kappa_t \omega}{H(t)} \sigma^D, \quad \beta(t) = -\omega \frac{1 - \kappa_t \omega^i}{H(t)} \sigma^D$$

$$\gamma(t) = -\omega \frac{(1 - \kappa_t \omega^i) \mu^D (T - t) - \omega^n \kappa_t \mu^\phi}{H(t)} \sigma^D, \quad \lambda(t, s) = \frac{\omega^i (\sigma^D)^2 (s - t)}{M(t)}.$$

Equilibrium exists if the backward Volterra equation,

$$A(t) = 1 - \omega^i \lambda(t, T) + \sigma^D \left(\int_t^T A(s) \beta(s) (1 - \omega^i \lambda(t, s)) ds \right), \quad A(T) = 1 \quad (1)$$

for the coefficient $A(\cdot)$ has a solution. This issue is addressed in the next lemma. ■

Lemma 24 *The unique solution of the backward Volterra equation is*

$$A(t) = \left(\frac{H(T)}{H(t)} \right)^\omega \left(\frac{M(T)}{M(t)} \right)^{1-\omega}$$

with $M(t) = (\omega^i)^2 H(t) + (\omega^n)^2 (\sigma^\phi)^2$ and $\omega = \omega^i + \omega^n$. Moreover, $A(t) > 0$ for $t \in [0, T]$.

Proof of Lemma 24. With $M(t) = (\omega^i)^2 H(t) + (\omega^n)^2 (\sigma^\phi)^2$, note that,

$$\begin{aligned} 1 - \omega^i \lambda(t, T) &= 1 - \frac{(\omega^i)^2 (\sigma^D)^2 (T - t)}{M(t)} = \frac{(\omega^i)^2 (\sigma^\zeta)^2 + (\omega^n)^2 (\sigma^\phi)^2}{M(t)} \equiv \frac{M(T)}{M(t)} \\ 1 - \omega^i \lambda(t, s) &= \frac{(\omega^i)^2 \left((\sigma^D)^2 (T - s) + (\sigma^\zeta)^2 \right) + (\omega^n)^2 (\sigma^\phi)^2}{M(t)} = \frac{M(s)}{M(t)}. \end{aligned}$$

Substituting in (1) and using the change of variables $C(t) = A(t)M(t)$ leads to,

$$\begin{aligned} A(t) &= 1 - \omega^i \lambda(t, T) + \sigma^D \left(\int_t^T A(s) \beta(s) (1 - \omega^i \lambda(t, s)) ds \right) \\ &= \frac{M(T)}{M(t)} + \sigma^D \left(\int_t^T A(s) \beta(s) \frac{M(s)}{M(t)} ds \right) \\ \iff A(t)M(t) &= M(T) + \sigma^D \left(\int_t^T A(s) \beta(s) M(s) ds \right) \\ \iff C(t) &= M(T) + \sigma^D \left(\int_t^T C(s) \beta(s) ds \right) \end{aligned}$$

subject to the boundary condition $C(T)M(T)$. Equivalently, $dC(t) = -\sigma^D C(t) \beta(t) dt$.

The solution is $C(t) = M(T) \exp \left(\sigma^D \int_t^T \beta(s) ds \right)$. Substituting,

$$\begin{aligned} \beta(t) &= -\frac{\omega}{H(t)} (1 - \kappa_t \omega^i) \sigma^D = -\frac{\omega}{H(t)} \left(1 - \frac{(\omega^i)^2 H(t)}{M(t)} \right) \sigma^D \\ &= -\omega \sigma^D \left(\frac{1}{H(t)} - \frac{(\omega^i)^2}{M(t)} \right) \end{aligned}$$

and performing the integration,

$$\begin{aligned} C(t) &= M(T) \exp \left(\omega \left(\log \left(\frac{H(T)}{H(t)} \right) - \log \left(\frac{M(T)}{M(t)} \right) \right) \right) \\ &= M(T) \left(\frac{H(T)}{H(t)} \right)^\omega \left(\frac{M(T)}{M(t)} \right)^{-\omega}. \end{aligned}$$

Substituting $C(t) = A(t)M(t)$ and rearranging leads to the formula stated. ■

Proof of Remark 5. Fix ω^n and let $\omega^i \rightarrow 0$ and $\omega^u \rightarrow 1 - \omega^n$. This yields $\kappa_t^{si} = \lambda^{si}(t, s) = 0$, $M^{si}(t) = (\omega^n)^2 (\sigma^\phi)^2$ and,

$$A^{si}(t) = \left(\frac{H(T)}{H(t)} \right)^{\omega^n}, \quad B^{si}(t) = \sigma^D \left(\int_t^T A^{si}(s) \alpha^{si}(s) ds \right) \quad (2)$$

$$F^{si}(t) = A^{si}(t) \mu^D (T - t) - \frac{(\sigma^D)^2}{\Gamma} \int_t^T A^{si}(s)^2 ds + \sigma^D \int_t^T A^{si}(s) \gamma^{si}(s) ds \quad (3)$$

$$\alpha^{si}(t) = \frac{\sigma^D}{H(t)}, \quad \beta^{si}(t) = -\omega^n \frac{\sigma^D}{H(t)}, \quad \gamma^{si}(t) = -\omega^n \frac{\mu^D (T - t)}{H(t)} \sigma^D. \quad (4)$$

The formulas stated follow. If, in addition, $\omega^n \rightarrow 0$, then $\beta^{si}(t) = \gamma^{si}(t) = M^{si}(t) = Z^{si} = 0$ and $A^{si}(t) = 1$. The stock price and return components announced follow. Note that $A(t) < A^{si}(t) < 1$ for $t < T$ and $A(T) = A^{si}(T) = 1$. Therefore, $\sigma_t^S < \sigma_t^{S,si} < \sigma_t^{S,si,0} = \sigma^D$ for $t < T$. In the limit, $\lim_{t \rightarrow T} \sigma_t^S = \lim_{t \rightarrow T} \sigma_t^{S,si} = \lim_{t \rightarrow T} \sigma_t^{S,si,0} = \sigma^D$. ■

Proof of Remark 6. Fix ω^n and let $\omega^i \rightarrow 1 - \omega^n$ and $\omega^u \rightarrow 0$. This yields

$$A^{su}(t) = \frac{H(T)}{H(t)}, \quad B^{su}(t) = \lambda^{su}(t, T) + \sigma^D \left(\int_t^T A^{su}(s) (\alpha^{su}(s) + \beta^{su}(s) \lambda^{su}(t, s)) ds \right) \quad (5)$$

$$F^{su}(t) = A^{su}(t) \mu^D (T - t) - \frac{(\sigma^D)^2}{\Gamma} \int_t^T A^{su}(s)^2 ds + \sigma^D \int_t^T A^{su}(s) \gamma^{su}(s) ds - \omega^n I^{su}(t) \mu^\phi \quad (6)$$

$$I^{su}(t) = \lambda^{su}(t, T) + \sigma^D \int_t^T A^{su}(s) \beta^{su}(s) \lambda^{su}(t, s) ds \quad (7)$$

$$\alpha^{su}(t) = \frac{1 - \kappa_t^{su}}{H(t)} \sigma^D, \quad \beta^{su}(t) = -\frac{1 - \kappa_t^{su}(1 - \omega^n)}{H(t)} \sigma^D, \quad \kappa_t^{su} = \frac{(1 - \omega^n) H(t)}{M^{su}(t)} \quad (8)$$

$$\gamma^{su}(t) = -\frac{(1 - \kappa_t^{su}(1 - \omega^n)) \mu^D(T - t) - \omega^n \kappa_t^{su} \mu^\phi}{H(t)} \sigma^D \quad (9)$$

$$\lambda^{su}(t, s) = \frac{(1 - \omega^n) (\sigma^D)^2 (s - t)}{M^{su}(t)} \quad (10)$$

and $M^{su}(t) = (1 - \omega^n)^2 H(t) + (\omega^n)^2 (\sigma^\phi)^2$. The formulas stated follow. If, in addition, $\omega^n \rightarrow 0$, then $\alpha^{su}(t) = \beta^{su}(t) = \gamma^{su}(t) = 0$, $\kappa_t^{su} = 1$ and,

$$A^{su,0}(t) = A^{su}(t) = \frac{H(T)}{H(t)}, \quad B^{su,0}(t) = I^{su,0}(t) = \lambda^{su,0}(t, T) = \frac{(\sigma^D)^2 (s - t)}{H(t)} \quad (11)$$

$$F^{su,0}(t) = A^{su}(t) \mu^D(T - t) - \frac{(\sigma^D)^2}{\Gamma} \int_t^T A^{su}(s)^2 ds \quad (12)$$

and $M^{su,0}(t) = H(t)$. This gives the formulas announced. Volatility rankings follows from $A^{su,0}(t) = A^{su}(t) < A(t)$ for all $t < T$ and $A(T) = A^{su}(T) = A^{su,0}(T) = 1$. ■

Proof of Corollary 7. The proof follows from Lemmas 25 and 26. ■

Proof of Corollary 8. The proof follows from Corollary 7 and Lemmas 25 and 26. ■

Proof of Corollary 9. The proof follows from Corollary 8 and Lemma 25. ■

Proof of Corollary 10. Differentiating with respect to the risk tolerance parameter gives the results. ■

Proof of Corollary 11. The results regarding the impact of s follows from Corollary 8 and Lemma 26. The results about μ^ϕ follow from the structure of the coefficient $\gamma(t)$. ■

The next auxiliary lemmas are used to derive comparative statics results. Proofs

are straightforward, but long and tedious. See Detemple, Rindisbacher and Truong (2014).

Lemma 25 *The following holds,*

$$\begin{aligned}
\frac{\partial H(t)}{\partial t} &= -(\sigma^D)^2 < 0, & \frac{\partial M(t)}{\partial t} &= (\omega^i)^2 \frac{\partial H(t)}{\partial t} < 0 \\
\frac{\partial \kappa_t}{\partial t} &= \frac{\omega^i (\omega^n)^2 (\sigma^\phi)^2}{M(t)^2} \frac{\partial H(t)}{\partial t} < 0, & \frac{\partial \lambda(t, s)}{\partial t} &= -\omega^i (\sigma^D)^2 \frac{M(s)}{M(t)^2} < 0 \\
\frac{\partial A(t)}{\partial t} &= -A(t) \left(\frac{\omega}{H(t)} + \frac{(1-\omega)(\omega^i)^2}{M(t)} \right) \frac{\partial H(t)}{\partial t} > 0 \\
\frac{\partial \alpha(t)}{\partial t} &= -\frac{(\omega^n)^2 (\sigma^\phi)^2 \omega \kappa_t + (1-\kappa_t \omega) M(t)}{M(t) H(t)^2} \frac{\partial H(t)}{\partial t} \sigma^D \geq 0 \iff \kappa_t^2 \leq \frac{1}{\omega^i \omega} \\
\frac{\partial \beta(t)}{\partial t} &= \omega \frac{(\omega^n)^4 (\sigma^\phi)^4 + 2(\omega^i)^2 H(t) (\omega^n)^2 (\sigma^\phi)^2}{M(t)^2 H(t)^2} \frac{\partial H(t)}{\partial t} \sigma^D < 0 \\
\partial \gamma(t) &= \frac{\omega \sigma^D}{H(t)^2} \left(\begin{array}{c} \frac{\partial \kappa_t}{\partial t} (\omega^i \mu^D (T-t) - \omega^n \mu^\phi) H(t) \\ -\omega^n \kappa_t \mu^\phi (\sigma^D)^2 + (1-\kappa_t \omega^i) \mu^D (\sigma^\zeta)^2 \end{array} \right) \\
\left\{ \begin{array}{l} \partial \gamma(t) > 0 \iff 0 \leq H(t) < H(t)^+ \\ \partial \gamma(t) < 0 \iff H^+ < H(t) \end{array} \right., & \quad H^+ = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad (13)
\end{aligned}$$

$$a = s^2 \left(s (\sigma^D)^2 \mu^\phi + (\sigma^\phi)^2 \mu^D \right), \quad b = -2s^2 (\sigma^\phi)^2 \mu^D (\sigma^\zeta)^2 \quad (14)$$

$$c = -(\sigma^\phi)^2 (\sigma^\phi)^2 \mu^D (\sigma^\zeta)^2, \quad s = \frac{\omega^i}{\omega^n} \quad (15)$$

$$\frac{\partial B(t)}{\partial t} = - \left(\frac{\omega^i (\sigma^D)^2}{M(t)} + \sigma^D \alpha(t) \right) A(t) < 0.$$

Lemma 26 *Let $s = \omega^i / \omega^n$ and $E(t) = s^2 H(t) + (\sigma^\phi)^2$. The following holds,*

$$\frac{\partial A(t)}{\partial s} = -(1-\omega) \left(\frac{H(T)}{H(t)} \right)^\omega \left(\frac{E(T)}{E(t)} \right)^{-\omega} \frac{2s (\sigma^\phi)^2 (\sigma^D)^2 (T-t)}{E(t)^2} < 0$$

$$\frac{\partial \alpha(t)}{\partial s} = -\frac{-s^2 H(t) + (2s+1) (\sigma^\phi)^2}{E(t)^2} \sigma^D \geq 0 \iff H(t) \geq \frac{2s+1}{s^2} (\sigma^\phi)^2$$

$$\frac{\partial \beta(t)}{\partial s} = \omega \frac{2s (\sigma^\phi)^2}{E(t)^2} \sigma^D > 0$$

$$\begin{aligned} \frac{\partial \gamma(t)}{\partial s} &= \omega \frac{2s (\sigma^\phi)^2 \mu^D (T-t) - \left(-s^2 H(t) + (\sigma^\phi)^2 \right) \mu^\phi}{E(t)^2} \sigma^D \geq 0 \\ \iff 2s (\sigma^\phi)^2 \mu^D (T-t) &\geq \left(-s^2 H(t) + (\sigma^\phi)^2 \right) \mu^\phi. \end{aligned}$$

$$\frac{\partial A(t)}{\partial \mu^\phi} = \frac{\partial B(t)}{\partial \mu^\phi} = \frac{\partial \alpha(t)}{\partial \mu^\phi} = \frac{\partial \beta(t)}{\partial \mu^\phi} = 0, \quad \frac{\partial \gamma(t)}{\partial \mu^\phi} = -\omega \frac{\omega^n \kappa_t}{H(t)} \sigma^D$$

$$\frac{\partial F(t)}{\partial \mu^\phi} = -\omega (\sigma^D)^2 \int_t^T s \frac{A(s)}{E(s)} ds - \omega^n I(t).$$

Proof of Corollary 12. Straightforward, but lengthy derivations lead to,

$$\frac{\partial m(t)}{\partial t} = \frac{2A(t) \partial A(t) \beta(t) - A(t)^2 \partial \beta(t)}{\beta(t)^2} \frac{\sigma_D}{\omega^u \Gamma} = \frac{A(t)^2}{\beta(t)} \left(2 \frac{\partial_t A(t)}{A(t)} - \frac{\partial \beta(t)}{\beta(t)} \right) \frac{\sigma_D}{\omega^u \Gamma}$$

$$\frac{\partial m(t)}{\partial s} = \frac{A(t)^2}{\beta(t)} \left(2 \frac{\partial_s A(t)}{A(t)} - \frac{\partial_s \beta(t)}{\beta(t)} \right) \frac{\sigma_D}{\omega^u \Gamma}$$

$$2 \frac{\partial_t A(t)}{A(t)} - \frac{\partial \beta(t)}{\beta(t)} = -2 \left(\omega - \frac{1}{2} \right) \frac{(\omega^n)^2 (\sigma^\phi)^2}{M(t) H(t)} \frac{\partial H(t)}{\partial t}$$

$$2 \frac{\partial_s A(t)}{A(t)} - \frac{\partial_s \beta(t)}{\beta(t)} = 2s (\sigma^\phi)^2 \frac{M(t) (\omega^n)^2 (\sigma^\phi)^2 \Pi(t) + (\omega^i)^2 (\sigma^\zeta)^2 H(t)}{E(t)^2 M(T) (\omega^n)^2 (\sigma^\phi)^2} > 0$$

where $\Pi(t) = (\sigma^\zeta)^2 + \omega (\sigma^D)^2 (T-t)$. ■

Proof of Proposition 16. The informed investors' optimization problem

$$U^i = \max_{N_t^i} E \left[X_T^i - \frac{1}{2\Gamma^i} \int_0^T d \langle X^i \rangle_t \middle| \mathcal{F}_0^G \right]$$

st

$$dX_t^i = N_t^i \sigma_t \left(\left(\theta_t + \theta_t^{G|m} (G) \right) dt + dW_t^G \right)$$

which is equivalent to (because dW_t^G is Brownian Motion with respect to dW_t^G)

$$\max_{N_t^i} E \left[x_0^i + \int_0^T N_t^i \sigma_t \left(\theta_t + \theta_t^{G|m}(G) \right) dt - \frac{1}{2\Gamma^i} \int_0^T (N_t^i)^2 \sigma_t^2 dt \right]$$

then his optimal number of share is

$$N_t^i = \Gamma^i \frac{\theta_t + \theta_t^{G|m}(G)}{\sigma_t}$$

The investor who chooses to self-managed his portfolio solves the problem

$$U^u(\Gamma^u) = \max_{N_t^u} E \left[X_T^u - \frac{1}{2\Gamma^u} \int_0^T d \langle X^u \rangle_t \middle| \mathcal{F}_0^m \right]$$

st

$$dX_t^u = N_t^u \sigma_t (\theta_t dt + dW_t)$$

Similarly to the informed investor's problem

$$N_t^u(\Gamma^u) = \Gamma^u \frac{\theta_t}{\sigma_t}$$

The active fund portfolio is the maximizer of

$$U^{i,i} = \max_{N_t^{u,i}} \mathbb{E}^i \left[\underbrace{\lambda (X_T^u - x_0^u)}_{\text{utility from compensation}} - \frac{1}{2q} \int_0^T d \langle X^{u,i} \rangle_t \middle| \mathcal{F}_0^G \right]$$

where q is the fund manager's risk taking level.

$$\begin{aligned} & \max_{N_t^{u,i}} \mathbb{E}^i \left[\underbrace{\lambda (X_T^u - x_0^u)}_{\text{utility from compensation}} - \frac{1}{2q} \int_0^T d \langle X^{u,i} \rangle_t \middle| \mathcal{F}_0^G \right] \\ & \max_{N_t^{u,i}} E \left[\lambda \int_0^T N_t^{u,i} \sigma_t \left(\theta_t + \theta_t^{G|m}(G) \right) dt - \frac{1}{2q} \int_0^T (N_t^{u,i})^2 \sigma_t^2 dt \right] \end{aligned}$$

$$N_t^{u,i} = \frac{\lambda q \left(\theta_t + \theta_t^{G|m}(G) \right)}{\sigma_t}$$

■

Proof of Proposition 18. The investors face 2 options: self-manage or delegate his wealth. The utility obtained from the first option is

$$\begin{aligned} U^u(\Gamma^u) &= E \left[X_T^u - \frac{1}{2\Gamma^u} \int_0^T d\langle X^u \rangle_t \middle| \mathcal{F}_0^m \right], N_t^u(\Gamma^u) = \Gamma^u \frac{\theta_t}{\sigma_t} \\ U^u(\Gamma^u) &= E \left[x_0^u + \int_0^T N_t^u(\Gamma^u) \sigma_t \theta_t dt - \frac{1}{2\Gamma^u} \int_0^T (N_t^u(\Gamma^u) \sigma_t)^2 \middle| \mathcal{F}_0^m \right], N_t^u(\Gamma^u) = \Gamma^u \frac{\theta_t}{\sigma_t} \\ U^u(\Gamma^u) &= E \left[x_0^u + \frac{\Gamma^u}{2} \int_0^T \theta_t^2 dt \middle| \mathcal{F}_0^m \right]. \end{aligned}$$

The utility obtained from the second option is

$$\begin{aligned} U^{u,i}(\Gamma^u) &= \mathbb{E} \left[(1-\lambda) X_T^{u,i} + \lambda x_0^u \middle| \mathcal{F}_0^m \right], N_t^{u,i} = \frac{\lambda q \left(\theta_t + \theta_t^{G|m}(G) \right)}{\sigma_t} \\ &= \mathbb{E} \left[(1-\lambda) \lambda q \int_0^T \left(\theta_t + \theta_t^{G|m}(G) \right)^2 + (1-\lambda) x_0^u + \lambda x_0^u \middle| \mathcal{F}_0^m \right]. \end{aligned}$$

Therefore, he will manage his wealth by himself iff

$$\begin{aligned} U^u(\Gamma^u) &\geq U^{u,i}(\Gamma^u) \\ x_0^u + \frac{\Gamma^u}{2} \mathbb{E} \left[\int_0^T \theta_t^2 dt \middle| \mathcal{F}_0^m \right] &\geq x_0^u + (1-\lambda) \lambda q \mathbb{E} \left[\int_0^T \left(\theta_t + \theta_t^{G|m}(G) \right)^2 \middle| \mathcal{F}_0^m \right] \\ \Gamma^u &\geq 2(1-\lambda) \lambda q \mathbb{E} \left[\int_0^T \left(\theta_t + \theta_t^{G|m}(G) \right)^2 \middle| \mathcal{F}_0^m \right] \bigg/ \mathbb{E} \left[\int_0^T \theta_t^2 dt \middle| \mathcal{F}_0^m \right]. \end{aligned}$$

■

Proof of Proposition 20. For each client the fund manager attracts with the

contract λ , he earn an extra utility

$$\begin{aligned} & \mathbb{E}^i \left[\underbrace{\lambda (X_T^u - x_0^u)}_{\text{utility from compensation}} - \frac{1}{2q} \int_0^T d \langle X^{u,i} \rangle_t \middle| \mathcal{F}_0^G \right] \\ = & \mathbb{E}^i \left[\lambda \int_0^T N_t^{u,i} \sigma_t \left(\theta_t + \theta_t^{G|m} (G) \right) dt - \frac{1}{2q} \int_0^T (N_t^{u,i})^2 \sigma_t^2 dt \right] \end{aligned}$$

where $N_t^{u,i} = \frac{\lambda q (\theta_t + \theta_t^{G|m} (G))}{\sigma_t}$

$$\frac{1}{2} \lambda^2 q \mathbb{E}^i \left[\int_0^T \left(\theta_t^{G|m} + \theta \right)^2 dt \middle| \mathcal{F}_0^G \right]$$

Because the mass of investors $F(\Gamma^{u,\lambda})$ is attracted by the actively managed funds, the manager choose the optimal investment management fee that maximizes his utility

$$\begin{aligned} & \max_{\lambda \in [0,1]} \omega^u F(\Gamma^{u,\lambda}) \frac{1}{2} \lambda^2 q \mathbb{E}^i \left[\int_0^T \left(\theta_t^{G|m} + \theta \right)^2 dt \middle| \mathcal{F}_0^G \right] \\ \iff & \max_{\lambda \in [0,1]} F(\Gamma^{u,\lambda}) \lambda^2. \end{aligned}$$

First order condition

$$2\lambda F(\Gamma^{u,\lambda}) + \lambda^2 f(\Gamma^{u,\lambda}) \frac{\partial \Gamma^{u,\lambda}}{\partial \lambda} = 0$$

we have

$$\Gamma^{u,\lambda} = \mathbb{T} = 2(1-\lambda) \lambda q \mathbb{E} \left[\int_0^T \left(\theta_t + \theta_t^{G|m} (G) \right)^2 \middle| \mathcal{F}_0^m \right] \bigg/ \mathbb{E} \left[\int_0^T \theta_t^2 dt \middle| \mathcal{F}_0^m \right]$$

then

$$\begin{aligned}
\frac{\partial \Gamma^{u,\lambda}}{\partial \lambda} &= 2(1-2\lambda) q \mathbb{E} \left[\int_0^T \left(\theta_t + \theta_t^{G|m}(G) \right)^2 \middle| \mathcal{F}_0^m \right] \bigg/ \mathbb{E} \left[\int_0^T \theta_t^2 dt \middle| \mathcal{F}_0^m \right] \\
&= 2(1-2\lambda) \frac{\Gamma^{u,\lambda}}{2(1-\lambda)\lambda} \\
&= \frac{1-2\lambda}{(1-\lambda)\lambda} \Gamma^{u,\lambda}.
\end{aligned}$$

Hence, the first order condition becomes

$$\begin{aligned}
2F(\Gamma^{u,\lambda}) &= f(\Gamma^{u,\lambda}) \frac{2\lambda-1}{(1-\lambda)} \Gamma^{u,\lambda} \\
\frac{2F(\Gamma^{u,\lambda})}{\Gamma^{u,\lambda} f(\Gamma^{u,\lambda})} &= -2 + \frac{1}{1-\lambda} \\
\lambda &= 1 - \frac{1}{2 \frac{F(\Gamma^{u,\lambda})}{\Gamma^{u,\lambda} f(\Gamma^{u,\lambda})} + 1}.
\end{aligned}$$

The second order condition is

$$\begin{aligned}
2\lambda F(\Gamma^{u,\lambda}) + \lambda^2 f(\Gamma^{u,\lambda}) \frac{\partial \Gamma^{u,\lambda}}{\partial \lambda} &= 0 \\
0 &> 2 \left(F(\Gamma^{u,\lambda}) + \lambda f(\Gamma^{u,\lambda}) \frac{\partial \Gamma^{u,\lambda}}{\partial \lambda} \right) \\
&\quad + 2\lambda f(\Gamma^{u,\lambda}) \left(\frac{\partial \Gamma^{u,\lambda}}{\partial \lambda} \right)^2 + \lambda^2 \left(f'(\Gamma^{u,\lambda}) \frac{\partial \Gamma^{u,\lambda}}{\partial \lambda} + f(\Gamma^{u,\lambda}) \frac{\partial^2 \Gamma^{u,\lambda}}{\partial \lambda^2} \right).
\end{aligned}$$

Substitute

$$\frac{\partial \Gamma^{u,\lambda}}{\partial \lambda} = \frac{1-2\lambda}{(1-\lambda)\lambda} \Gamma^{u,\lambda} \text{ and } \frac{\partial^2 \Gamma^{u,\lambda}}{\partial \lambda^2} = -4 \frac{\Gamma^{u,\lambda}}{2(1-\lambda)\lambda}$$

the second order condition becomes

$$0 > 2F(\Gamma^{u,\lambda}) + (\lambda f(\Gamma^{u,\lambda}) + \lambda^2 f'(\Gamma^{u,\lambda})) \frac{1-2\lambda}{(1-\lambda)\lambda} \Gamma^{u,\lambda} \\ + 2\lambda f(\Gamma^{u,\lambda}) \left(\frac{1-2\lambda}{(1-\lambda)\lambda} \Gamma^{u,\lambda} \right)^2 - 2\lambda^2 f(\Gamma^{u,\lambda}) \frac{\Gamma^{u,\lambda}}{(1-\lambda)\lambda}.$$

■

Proof of Proposition 21. Let rewrite the equation (4.1) as follows

$$qr = \frac{F(\Gamma^{u,\lambda})}{\lambda(2\lambda-1)f(\Gamma^{u,\lambda})} \quad (16) \\ \Gamma^{u,\lambda} = 2(1-\lambda)\lambda qr$$

where $r = \mathbb{E} \left[\int_0^T \left(\theta_t + \theta_t^{G|m}(G) \right)^2 \middle| \mathcal{F}_0^m \right] / \mathbb{E} \left[\int_0^T \theta_t^2 dt \middle| \mathcal{F}_0^m \right]$. First, the manager's risk taking level cannot approach 0 because the equation (16) must hold. That means $0 = \lim_{q \rightarrow 0} \frac{F(\Gamma^{u,\lambda})}{\lambda(2\lambda-1)f(\Gamma^{u,\lambda})}$ but $\frac{F(\Gamma^{u,\lambda})}{\lambda(2\lambda-1)f(\Gamma^{u,\lambda})} > 0$ for all parameters. Thus, there exists a lower bound q_{\min} for the manager's risk taking level. Similarly, the manager's risk taking level cannot approach ∞ because in order for (16) to hold, $\lambda \rightarrow \frac{1}{2}$ then $F(\Gamma^{u,\lambda}) \rightarrow 0$ or the active fund collapses. This is obviously sub-optimal for the fund manager if he could choose the level of risk-taking. Thus, there exists an upper bound q_{\max} for the manager's risk-taking level. In other words, the necessary condition for an equilibrium to exist is q is bounded by $[q_{\min}, q_{\max}]$ where the form of q_{\min} and q_{\max} depend on the distribution F , the risk tolerance band $[\underline{\Gamma}^u, \bar{\Gamma}^u]$ and the signal G . Let the curve \mathcal{C} be the set of pairs $(\lambda, \Gamma^{u,\lambda})$ such that $\frac{F(\Gamma^{u,\lambda})}{\lambda(2\lambda-1)f(\Gamma^{u,\lambda})} = \frac{\Gamma^{u,\lambda}}{2(1-\lambda)\lambda}$. This set is continuous because $F(\cdot)$ is continuous. Then along this curve, the value $\frac{F(\Gamma^{u,\lambda})}{\lambda(2\lambda-1)f(\Gamma^{u,\lambda})}$ is continuous, also $\max_{(\lambda, \Gamma^{u,\lambda}) \subseteq \mathcal{C}} \frac{F(\Gamma^{u,\lambda})}{\lambda(2\lambda-1)f(\Gamma^{u,\lambda})} = r q_{\max}$ and $\min_{(\lambda, \Gamma^{u,\lambda}) \subseteq \mathcal{C}} \frac{F(\Gamma^{u,\lambda})}{\lambda(2\lambda-1)f(\Gamma^{u,\lambda})} = r q_{\min}$. By intermediate value theorem, for any $q \in [q_{\min}, q_{\max}]$, there exists a pair $(\lambda, \Gamma^{u,\lambda}) \subseteq \mathcal{C}$, such that $\frac{F(\Gamma^{u,\lambda})}{\lambda(2\lambda-1)f(\Gamma^{u,\lambda})} = qr$. The

condition (4.3) that ensures the threshold risk tolerance belongs to its domain $[\underline{\Gamma}^u, \bar{\Gamma}^u]$ is automatically satisfied because of the definitions of q_{\min} and q_{\max} . ■

Proof of Proposition 22. Even though the quantities θ_t and $\theta_t^{G|m}(G)$ are endogenous in general equilibrium, the ratio $r = \frac{\mathbb{E}\left[\int_0^T \left(\theta_t + \theta_t^{G|m}(G)\right)^2 \middle| \mathcal{F}_0^m\right]}{\mathbb{E}\left[\int_0^T \theta_t^2 dt \middle| \mathcal{F}_0^m\right]}$ is still bounded. Easy to see that it is bounded below by 1. To show that it is bounded above, we can note that investors' risk tolerance range is bounded, so the market price of risk θ_t^2 is always bounded below (In fact, the market price of risk reaches 0 only if the public overall risk tolerance approaches ∞ , i.e. they are risk neutral) and the private information price of risk $\theta_t^{G|m}(G)$ is bounded above because of the noise ζ contained in the signal G ($\theta_t^{G|m}(G)$ goes to ∞ iff the private signal has no noise, i.e $\zeta = 0$) Therefore, same argument as in Proof of Proposition 21 applies ■

Proof of Proposition 23. We have shown in the previous section that any investment management fee λ will split the investor population into two separate groups, the group with high risk tolerance (i.e. $\Gamma^u \geq \Gamma^{u,\lambda}$) will manage their own portfolios and the remaining investors with low risk tolerance (i.e. $\Gamma^u \leq \Gamma^{u,\lambda}$) will invest in the active managed fund.

Define

$$F(\Gamma) \equiv \int_{\underline{\Gamma}^u}^{\Gamma} f(\Gamma^u) d\Gamma^u \text{ and } E(\Gamma) \equiv \int_{\Gamma}^{\bar{\Gamma}^u} \Gamma^u f(\Gamma^u) d\Gamma^u.$$

First, we solve for the equilibrium prices where the investment fee λ and the threshold risk tolerance Γ are determined exogenously. Hence, the total demand of shares from the first group is

$$\begin{aligned} N_t^1 &= \omega^u \int_{\underline{\Gamma}^u}^{\Gamma} N^{u,i} f(\Gamma^u) d\Gamma^u = \int_{\underline{\Gamma}^u}^{\Gamma} \lambda q \frac{\theta_t + \theta_t^{G|m}(G)}{\sigma_t} f(\Gamma^u) d\Gamma^u \\ &= \omega^u F(\Gamma) \lambda q \frac{\theta_t + \theta_t^{G|m}(G)}{\sigma_t}. \end{aligned}$$

While the total demand of shares from the second group is

$$N_t^2 = \omega^u \int_{\Gamma}^{\bar{\Gamma}^u} N^u(\Gamma^u) f(\Gamma^u) d\Gamma^u = \omega^u \int_{\Gamma}^{\bar{\Gamma}^u} \frac{\Gamma^u \theta_t}{\sigma_t} f(\Gamma^u) d\Gamma^u = \omega^u E(\Gamma) \frac{\theta_t}{\sigma_t}.$$

The representative informed investor's and the noise trader's demand are

$$\omega^i N^i = \omega^i \Gamma^i \frac{\theta_t + \theta_t^{G|m}(G)}{\sigma_t} \text{ and } \omega^n N^n = \omega^n \frac{\theta_t + \theta_t^{G|m}(\phi)}{\sigma_t}.$$

The market clearing condition imposes

$$N_t^{\text{demand}} = N_t^{\text{supply}}$$

$$\omega^u F(\Gamma) \lambda q \frac{\theta_t + \theta_t^{G|m}(G)}{\sigma_t} + \omega^u E(\Gamma) \frac{\theta_t}{\sigma_t} + \omega^i \Gamma^i \frac{\theta_t + \theta_t^{G|m}(G)}{\sigma_t} + \omega^n \Gamma^n \frac{\theta_t + \theta_t^{G|m}(\phi)}{\sigma_t} = 1.$$

Conjecture that the function $\theta_t^{G|m}(x)$ is affine function of x , then the market clearing condition gives

$$\theta_t = \frac{\sigma_t - \theta_t^{G|m}((\omega^u \lambda q F(\Gamma) + \omega^i \Gamma^i) G + \omega^n \Gamma^n \phi)}{\omega^u \frac{\lambda}{q} F(\Gamma) + \omega^u E(\Gamma) + \omega^i \Gamma^i + \Gamma^n \omega^n}.$$

Hence, information revealed in equilibrium includes the noisy translation of the private signal $Z \equiv (\omega^u \lambda q F(\Gamma) + \omega^i \Gamma^i) G + \omega^n \Gamma^n \phi$.

Define $w_u \equiv \omega^u E(\Gamma)$, $w_i \equiv (\omega^u \lambda q F(\Gamma) + \omega^i \Gamma^i)$, $w_n \equiv \omega^n \Gamma^n$, $\mathbb{W} = w_u + w_i + w_n$ then $Z = w_i G + w_n \phi$ and

$$\theta_t = \frac{\sigma_t - \theta_t^{G|m}(Z; w_i, w_n)}{\mathbb{W}}.$$

We have $\mathcal{F}_t^m \supseteq \mathcal{F}_t^{Z,D}$.

The new Brownian motion with respect to the public information filtration \mathcal{F}_t^m is defined as $dW_t^m \equiv dW_t^D - \theta_t^{m|D} dt$ and the new Brownian motion with respect to the private information filtration \mathcal{F}_t^G is defined as $dW_t^G \equiv dW_t^m - \theta_t^{G|m} dt$. Suppose that

$\mathcal{F}_t^m = \mathcal{F}_t^{Z,D}$, we obtain

$$\theta_t^{m|D} = \frac{w_i \sigma_D}{w_i^2 H_t + w_n^2 \sigma_\phi^2} (Z - w_i (D_t + \mu_D (T - t)))$$

$$\theta_t^{G|m} (G) = \frac{\sigma_D}{H_t} (G - D_t - \mu_D (T - t)) - \frac{w_i \sigma_D}{M_t} (Z - w_i (D_t + \mu_D (T - t))).$$

Hence, $\theta_t^{G|m} (x)$ is affine in x , as conjectured.

Compute $\theta_t^{G|m} (Z; w_i, w_n)$:

$$\begin{aligned} \theta_t^{G|m} (Z; w_i, w_n) &= w_i \theta_t^{G|m} (G) + w_n \theta_t^{G|m} (\phi) \\ &= \frac{\sigma_D}{H_t} Z + w_i \left(\begin{array}{c} \frac{\sigma_D}{H_t} (-D - \mu_D (T - t)) \\ -\frac{w_i \sigma_D}{M_t} (Z - w_i (D + \mu_D (T - t))) \end{array} \right) \\ &\quad + w_n \left(\begin{array}{c} \frac{\sigma_D}{H_t} (-D - \mu_D (T - t)) \\ -\frac{w_i \sigma_D}{M_t} (Z - w_i (D + \mu_D (T - t))) \end{array} \right) \\ &= \alpha(t) Z + \beta(t) D_t + \gamma_t. \end{aligned}$$

The informational content in $\theta_t^{G|m} (Z; w_i, w_n)$ is indeed equivalent to Z provided $\alpha(t) \neq 0$. The pair (Z, D) is a sufficient statistic for the PIPR. We still need to verify that the pair is a sufficient statistic for the rest of the equilibrium as well.

Assume the public investor use $F^{D,Z}$ to forecast future dividend and assess the market price of risk θ_t as follow

$$\theta_t = \frac{\sigma_t - \theta_t^{G|m} (w^i G + w^n \phi)}{\mathbb{W}} = \frac{\sigma_t - (\alpha(t) Z + \beta(t) D_t + \gamma_t)}{\mathbb{W}}$$

where

$$\begin{aligned} \alpha(t) &\equiv \sigma_D \frac{M_t - w_i H_t (w_i + w_n)}{H_t M_t} \\ \beta(t) &\equiv \sigma_D (-M_t + w_i^2 H_t) \frac{w_i + w_n}{H_t M_t} \\ \gamma(t) &\equiv \sigma_D \mu_D (-M_t + w_i^2 H_t) (w_i + w_n) \frac{T - t}{H_t M_t} \end{aligned}$$

and the expected stock return μ_t is

$$\mu_t = \frac{\sigma_t^2 - \sigma_t (\alpha(t) Z + \beta(t) D_t + \gamma_t)}{\mathbb{W}}.$$

Assuming the volatility coefficients are functions of time, it follows that the stock price will be

$$\begin{aligned} S_t &= D_t + \mu_D (T - t) - \frac{1}{\mathbb{W}} \int_t^T \sigma_s^2 ds + \frac{1}{\mathbb{W}} \int_t^T \sigma_s (\alpha_s Z + \gamma_s) ds \\ &\quad + \mathbb{E} \left[\int_t^T \sigma^D dW_s^D \middle| \mathcal{F}_t^{Z,D} \right] + \frac{1}{\mathbb{W}} \int_t^T \sigma_s \beta_s \mathbb{E} [D_s | \mathcal{F}_t^{Z,D}] ds \end{aligned}$$

where $\mathbb{E} \left[\int_t^T \sigma^D dW_s^D \middle| \mathcal{F}_t^{Z,D} \right]$ is equal to

$$\mathbb{E} \left[\int_t^T \sigma^D dW_s^D \middle| \mathcal{F}_t^{Z,D} \right] = \chi(t, T) Z - w^i \chi(t, T) D_t - \chi(t, T) (w^i \mu^D (T - t))$$

and $\mathbb{E} [D_s | \mathcal{F}_t^{Z,D}]$ is equal to

$$\mathbb{E} [D_s | \mathcal{F}_t^{Z,D}] = (D_t + \mu^D (T - t)) (1 - w^i \chi(t, s)) + \chi(t, s) Z$$

with

$$\chi(t, s) \equiv \frac{w_i \sigma_D^2 (s - t)}{M_t}$$

Combining the above expressions, we obtain the stock price is

$$S_t = A_t D_t + B_t Z + F_t$$

where

$$A_t \equiv 1 - w_i \chi(t, T) + \frac{1}{\mathbb{W}} \int_t^T \sigma_s \beta_s (1 - w_i \chi(t, s)) ds$$

$$\begin{aligned}
B_t &\equiv \chi(t, T) + \frac{1}{\mathbb{W}} \int_t^T \sigma_s (\alpha_s + \beta_s \chi(t, s)) ds \\
F_t &= \mu_D (T - t) - \frac{1}{\mathbb{W}} \int_t^T \sigma_s^2 ds + \frac{1}{\mathbb{W}} \int_t^T \sigma_s \gamma_s ds \\
&\quad - \chi(t, T) (w_i \mu_D (T - t)) \\
&\quad + \frac{1}{\mathbb{W}} \int_t^T \sigma_s \beta_s ((\mu_D (T - t)) (1 - w_i \chi(t, s))) ds
\end{aligned}$$

From Ito's lemma , the stock volatility is $\sigma_t = A_t \sigma_D$, then A_t , and B_t and F_t can be rewritten as

$$\begin{aligned}
A_t &= 1 - w_i \chi(t, T) + \frac{\sigma_D}{\mathbb{W}} \int_t^T A_s \beta_s (1 - w_i \chi(t, s)) ds \\
B_t &= \chi(t, T) + \frac{\sigma_D}{\mathbb{W}} \int_t^T A_s (\alpha_s + \beta_s \chi(t, s)) ds \\
F_t &= A_t \mu_D (T - t) - \frac{\sigma_D^2}{\mathbb{W}} \int_t^T A_s^2 ds + \frac{\sigma_D}{\mathbb{W}} \int_t^T A_s \gamma_s ds
\end{aligned}$$

Therefore, the stock volatility is deterministic as conjectured, and the pair (Z, D) is a sufficient statistic for the stock price. This confirms $\mathcal{F}_t^m = \mathcal{F}_t^{Z, D}$

The equilibrium exists if the backward Volterra equation

$$A_t = 1 - w_i \chi(t, T) + \frac{\sigma_D}{\mathbb{W}} \int_t^T A_s \beta_s (1 - w_i \chi(t, T)) ds$$

for the coefficient $A(\cdot)$ has a solution. The next Lemma gives

$$A_t = \left(\frac{H_T}{H_t} \right)^{\frac{w}{\mathbb{W}}} \left(\frac{M_T}{M_t} \right)^{1 - \frac{w}{\mathbb{W}}}$$

Therefore, given any pair (λ, Γ) , the equilibrium coefficients are $\sigma_t^{\mathcal{E}(\lambda, \Gamma)}$, $\theta_t^{\mathcal{E}(\lambda, \Gamma)}$ and

$\theta_t^{G|m;\mathcal{E}(\lambda,\Gamma)}$. In competitive NREE, the fund manager and investors' actions cannot affect the equilibrium prices. The equilibrium investment management fee λ^* and the threshold risk tolerance Γ^{u,λ^*} are the fixed points of the system of two equations

$$\begin{aligned}\lambda &= 1 - \frac{1}{2 \frac{F(\Gamma^{u,\lambda})}{\Gamma^{u,\lambda} f(\Gamma^{u,\lambda})} + 1} \\ \Gamma^{u,\lambda} &= 2(1-\lambda) \lambda q \mathbb{E} \left[\int_0^T \left(\theta_t + \theta_t^{G|m}(G) \right)^2 \middle| \mathcal{F}_0^m \right] \bigg/ \mathbb{E} \left[\int_0^T \theta_t^2 dt \middle| \mathcal{F}_0^m \right]\end{aligned}$$

and

$$\frac{1}{2} < \lambda^* < \max_{x \in [\underline{\Gamma}^u, \bar{\Gamma}^u]} \left(1 - \frac{1}{2 \frac{F(x)}{\Gamma^{u,\lambda} f(x)} + 1} \right).$$

■

Lemma 27 *The unique solution of the backward Volterra equation is*

$$A_t = \left(\frac{H_T}{H_t} \right)^{\frac{w}{\mathbb{W}}} \left(\frac{M_T}{M_t} \right)^{1 - \frac{w}{\mathbb{W}}}.$$

Moreover, $A(t) > 0$ for $t \in [0, T]$.

Proof. The equation

$$A_t = 1 - w_i \chi(t, T) + \frac{\sigma_D}{\mathbb{W}} \int_t^T A_s \beta_s (1 - w_i \chi(t, T)) ds$$

can be written as

$$A_t M_t = M_T + \frac{\sigma_D}{\mathbb{W}} \int_t^T A_s \beta_s M_s ds$$

Define $C_t \equiv A_t M_t$ with the boundary condition $C_T = M_T$ and $dC_t = -\frac{\sigma_D}{\mathbb{W}} C_t \beta_t dt$

where

$$\begin{aligned}
 C_t &= M(T) \exp \left(\frac{\sigma_D}{\mathbb{W}} \int_t^T \beta_s ds \right) \\
 &= M(T) \exp \left(\frac{1}{\mathbb{W}} \int_t^T \left(-\frac{\sigma_D^2}{H_s} + \frac{\sigma_D^2 \omega_i^2}{M_s} \right) ds \right)
 \end{aligned}$$

Hence the solution can be written as

$$A_t = \left(\frac{H_T}{H_t} \right)^{\frac{w}{\mathbb{W}}} \left(\frac{M_T}{M_t} \right)^{1 - \frac{w}{\mathbb{W}}}$$

■

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